REPORT ON STABLE GROUPS

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The model theory of first order stable groups is well developed, see [3, 4]. The aim for the group working on stable groups was to generalize some of the ideas about stable groups to the setting of metric structures. In particular, we concentrated on:

0.1. **Question.** *Let $(G, \cdot)$ be an ω-stable metric group. Is every type-definable subgroup of G definable?*

Some tools from stable group theory are easy to generelize. For example the existence of generic types in the setting of metric structures is proved in [1].

0.2. **Example.** Let $(X, \mathcal{B}, \mu)$ be an atomless probability space and let $\widehat{B}$ be the corresponding measure algebra of events. The structure $(\widehat{B}, \cup, \cap, ^c, \ , \ , \mu, d)$ is ω-stable and it has a definable group operation: the symmetric difference. Entropy, a notion from probability theory, is an additive rank with values in well defined for types and it characterizes non-forking. In the structure $(\widehat{B}, \triangle, \dots)$ every type-definable subgroup is definable (see [2]). The generics correspond to elements of maximal entropy.

Note that in the example above for $a, b, c \in \widehat{B}$ and $d(a, b) = d(a \cdot c, b \cdot c)$. From now on, we assume that the group operation is distance preserving. That is, for $(G, \cdot)$ a stable group, we require that for $a, b, c \in G$, $d(a \cdot c, b \cdot c) = d(a, b)$.

0.3. **Example.** Let σ be a generic automorphism of the structure $\widehat{B}$. The structure $(\widehat{B}, \cup, \cap, ^c, \ , \ , \mu, \sigma, d)$ is not ω-stable, but it is ω-stable up to perturbations of the automorphism. Let $Fix(\sigma) = \{a \in \widehat{B} : \sigma(a) = a\}$. Clearly $Fix(\sigma)$ is a type-definable subgroup. The type "$\sigma(x) = x \cup \ \mu(x) = 1/2$" can be omited and thus $Fix(\sigma)$ is not definable.

The previous example shows that the question of definability of type-definable subgroups has a negative answer when we allow perturbations.

0.4. **Lemma.** *Let $(G, \cdot, \dots)$ be a metric group and let H be a type-definable normal subgroup. Then the quotient group $(G/H, \cdot)$ is a metric structure.*

Proof. First we define an induced metric on the quotient. For $a, b \in G$, let $d'(aH, bH) = \inf\{d(ah_1, bh_2) : h_1, h_2 \in H\}$. Note that $d'(aH, bH) = d(b^{-1}a, H)$ and thus $d'(aH, bH) = 0$ iff $aH = bH$. It is easy to see that d' satisfies the triangle inequality and that $(G/H, d')$ is complete. □

One of the main obstacles in proving results about stable groups is that there is no single global rank that captures forking. In the ω-stable case, there is a family of ordinal valued Morley ranks RM_ϵ for every $\epsilon > 0$ and in the superstable case there is a family of Lascar ranks SU_ϵ for every $\epsilon > 0$ also with ordinal values. In these settings, when a type forks, there is $\epsilon > 0$ such that the ϵ-rank decreases, but as we consider forking chains, the values ϵ may vary. On the other hand, for $(G, \cdot)$ is an ω-stable group, a type $p \in S_1(G)$ is generic if and only if it has maximal RM_ϵ for every ϵ; and the quotient G/G^0 is a compact group.

We were also missing more examples of metric groups. In Example 0.2 every element has order 2. Ofcourse, every compact metric group is stable and so is its product with the measure algebra of a probability space. Is there a way to build examples of metric non-compact groups from compact metric groups?

0.5. **Example.** Consider S^1, the unit circle, where we measure the distance between two points as the angle between them. Consider now $G = \Pi_{i \in \omega} S^1$. With the product topology G is again a compact group. Instead, let us define, for $f, g \in G$, $d(f, g) = \sup_{n \in \omega} d(f(n), g(n))$. The structure $(G, \cdot, d)$ is a metric group, for $f, g, h \in G$, $d(f, g) = d(f \cdot h, g \cdot h)$.

Claim The structure $(G, \cdot, d)$ is not stable.

Let $f_n \in G$ be such that $f_n(n) = \pi$, $f_n(m) = \pi/4$ for $n \neq m$. Let $g_n(k) = 0$ for $k \leq n$, $g_n(m) = \pi/4$ otherwise. Then $d(f_n, g_m) = \pi$ for $n < m$ and $d(f_n, g_m) = \pi/4$ for $n > m$. The sequence $\{(f_n, g_n) : n \in \omega\}$ shows that $(G, \cdot, d)$ has the order property.

The argument above also works for any other compact group G that contains a *triangle*, three points $a, b, c \in G$ such that $d(a, b) > d(a, c)$, $d(a, b) > d(b, c)$

0.6. **Definition.** A complete theory T is said to be *truly continuous* if for every $\mathcal{M} \models T$, the set $\{(x, y) : x, y \in M, x \neq y\}$ in NOT closed.

0.7. **Lemma.** *There is no truly continuous field.*

Proof. Assume that F is an $\aleph_1$-saturated continuous field. Then $\{(x, y) : x \neq y\} = \{(x, y) : \exists z(x-y)z = 1\}$ is a closed set and thus $Th(F)$ is not truly continuous. $\square$

Since there are no truly metric fields, one expects a modular behavior among truly metric groups.

0.8. **Conjecture.** Let $(G, \cdot, \dots, d)$ be a metric group and let $T = Th(G, \cdot, \dots, d)$. Assume that T is truly continuous. Then $(G, \cdot, \dots)$ is abelian-by-compact.

References

[1] Itay Ben-Yaacov, *Thickness, and a categoric view of type-space functors*, Fundamenta Mathematicae, volume 179 (2003), 199-224.

[2] Alexander Berenstein, *Definable subgroups of measure algebras*, to appear in Archives for Mathematical Logic.

[3] Bruno Poizat, *Stable groups*, Mathematical Surveys and Monographs, American mathematical Society, 2001.

[4] Frank Wagner, *Stable groups*, London Mathematical Society Lecture Note Series 240, Cambridge University Press, 1997.