A hastily prepared introduction to perturbations

Itaï Ben-Yaacov

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Perturbations of metric structures

- Changing a classical structure (without changing the underlying set) consists of changing some predicate from "True" to "False" (or *vice versa*). $\Longrightarrow$ Changing by "little" = not changing at all.

- A continuous predicate/function can be changed by a little, e.g., by no more than $\varepsilon > 0$. $\Longrightarrow$ Notions of small perturbations of metric structures.

Formal definition [Ben]

Definition. A *perturbation system* $\mathfrak{p}$ for T consists of a family of $[0, \infty]$-valued metrics $d_{\mathfrak{p},n}$ on $\mathrm{S}_n(T)$, such that:

TM Each $d_{\mathfrak{p},n}$ is lower semi-continuous (i.e., $(\mathrm{S}_n(T), d_{\mathfrak{p}})$ is a *topometric* space).

INV $d_{\mathfrak{p},n}$ is invariant under permutations of n.

EXT Let $\pi \colon \mathrm{S}_{n+1}(T) \to \mathrm{S}_n(T)$, $p \in \mathrm{S}_n(T)$, $q \in \mathrm{S}_{n+1}(T)$:

$$d_{\mathfrak{p},n}(p, \pi(q)) = d_{\mathfrak{p},n+1}(\pi^{-1}(p), q)$$

UC If $b \neq c$: $\quad d_{\mathfrak{p},2}(\mathrm{tp}(aa), \mathrm{tp}(bc)) = \infty$.

Definition. A bijection $f \colon M \to N$ is a $\mathfrak{p}(r)$-*perturbation* if for all $\bar{a} \in M$:

$$d_{\mathfrak{p}}(\mathrm{tp}(f(\bar{a})), \mathrm{tp}(\bar{a})) \leq r.$$

The set of all $\mathfrak{p}(r)$-perturbations is denoted: $\mathrm{Pert}_{\mathfrak{p}(r)}(M, N)$.

By **UC** a perturbation is always uniformly continuous.

Fact. *Let $\bar{a} \in M$, $\bar{b} \in N$. TFAE:*

- $d_{\mathfrak{p}}(\mathrm{tp}(\bar{a}), \mathrm{tp}(\bar{b})) \leq r$.
- $\exists\big(M' \succeq M, N' \succeq N, \theta \in \mathrm{Pert}_{\mathfrak{p}(r)}(M', N')\big)(\theta(\bar{a}) = \bar{b})$.

This allows us to specify a perturbation system $\mathfrak{p}$ by specifying $\mathrm{Pert}_{\mathfrak{p}(r)}(M, N)$ for all M, N, r (they must satisfy some conditions...)

Examples

Example (Trivial perturbation system: id). $d_{\mathrm{id}}(p,q) = \begin{cases} 0 & p = q \\ \infty & p \neq q \end{cases}$; $\mathrm{Pert}_{\mathrm{id}(r)}(M,N) = \mathrm{Iso}(M,N)$.

Example (Banach Mazur distance). T = Banach spaces (with no additional structure). $\theta \in \mathrm{Pert}_{BM(r)}(E,F)$ if $\theta\colon E \to F$ is a linear bijection and:

$$\forall v \in E \qquad \|v\|e^{-r} \leq \|\theta(v)\| \leq \|v\|e^{r}.$$

Example (Perturbation of a new symbol). T = an $\mathcal{L}$-theory, $\mathcal{L}' = \mathcal{L} \cup \{P(\bar{x})\}$, $\mathfrak{p}$ a perturbation system for T. $\mathfrak{p}_P = \mathfrak{p}$ + perturbation of P:

$$\theta \in \mathrm{Pert}_{\mathfrak{p}_P(r)}((M,P),(N,P)) \iff \begin{cases} \theta \in \mathrm{Pert}_{\mathfrak{p}(r)}(M,N), \\ \text{and for all } \bar{b} \in M : \\ \qquad |P^M(\bar{b}) - P^N(\theta(\bar{b}))| \leq r \end{cases}$$

Same can be done with a finite tuple $\bar{P}$ of new symbols. A function symbol $f(\bar{x})$ can be replaces with $G_f(\bar{x},y) = d(f(\bar{x}),y)$. $\rightsquigarrow$ Perturbations of the automorphism in (M,σ). $\rightsquigarrow$ Perturbations of parameters of types.

- When doing "model theory up to $\mathfrak{p}$-perturbations" we mustn't forget the standard metric d on types. We merge d and $d_{\mathfrak{p}}$ by allowing to perturb the structure *and* move the realisations.

- We define $\tilde{d}_{\mathfrak{p}}(p,q)$ as the minimum r for which there are $M \vDash p(\bar{e})$, $N \vDash q(\bar{f})$ and $\theta \in \mathrm{Pert}_{\mathfrak{p}r}(M,N)$ such that: $(\forall a \in M)\big(|d^M(a,e_i) - d^N(\theta(a),f_i)| \leq r\big)$. This is a natural common coarsening of d and $d_{\mathfrak{p}}$ on $\mathrm{S}_n(T)$.

- Note that if $M \vDash p(\bar{e})$, $N \vDash q(\bar{f})$ then:

$$\tilde{d}_{\mathfrak{p},n}(p,q) = d_{\mathfrak{p}_{\bar{c}},0}(\mathrm{Th}_{\mathcal{L}(\bar{c})}(M,\bar{e}), \mathrm{Th}_{\mathcal{L}(\bar{c})}(N,\bar{f}))$$

- Finally, for $p,q \in \mathrm{S}_n(\bar{a})$: $\mathfrak{p}_{\bar{a}}$ allows to move the parameters $\bar{a}$, and $\tilde{d}_{\mathfrak{p}_{\bar{a}}}(p,q)$ is the minimal distance we need to "travel", moving parameters and realisations (and perturbing the underlying structure), to get from p to q.

Categoricity up to perturbation

Definition.

- Two structures are $\mathfrak{p}$-*isomorphic*, $M \simeq_{\mathfrak{p}} N$, if for all $\varepsilon > 0$ there exists an ε-perturbation $\theta \in \mathrm{Pert}_{\mathfrak{p}(\varepsilon)}(M,N)$.

- T is $\mathfrak{p}$-λ-*categorical* if $M, N \vDash T$ and $\|M\| = \|N\| = \lambda$ imply $M \simeq_{\mathfrak{p}} N$.

Perturbed Ryll-Nardzewski

Recall:

Definition. A point $x \in X$ is *d-isolated* if for all $r > 0$: $x \in B(x,r)^\circ$ (i.e., the metric and the topology coincide at x).

It is *weakly d-isolated* if we only have $B(x,r)^\circ \neq \varnothing$ for all $r > 0$.

Theorem. *Let T be a countable complete theory. TFAE:*

- *T is $\mathfrak{p}$-$\aleph_0$-categorical.*
- *For all finite $\bar{a}$, every point in the topometric space $(\mathrm{S}_1(\bar{a}), \tilde{d}_{\mathfrak{p}_{\bar{a}}})$ is weakly $\tilde{d}_{\mathfrak{p}_{\bar{a}}}$-isolated.*
- *For all finite, there is a $\tilde{d}_{\mathfrak{p}_{\bar{a}}}$-dense subset of $(\mathrm{S}_1(\bar{a}), \tilde{d}_{\mathfrak{p}_{\bar{a}}})$ consisting of $\tilde{d}_{\mathfrak{p}_{\bar{a}}}$-isolated points.*

Corollary (Sufficient condition, no parameters)**.** *Assume T is countable, complete, and for every $n < \omega$ each point in $(\mathrm{S}_n(T), \tilde{d}_{\mathfrak{p}})$ is $\tilde{d}_{\mathfrak{p}}$-isolated. Then T is $\mathfrak{p}$-$\aleph_0$-categorical.*

Corollary (Transfer to a reduct)**.** *Assume T is countable, complete. Let $\mathcal{L}' = \mathcal{L} \cup \{\bar{P}\}$, T' an $\mathcal{L}'$-completion of T, $\mathfrak{p}' = \mathfrak{p}_{\bar{P}}$. If for every $n < \omega$ each point in $(\mathrm{S}_n(T'), \tilde{d}_{\mathfrak{p}'})$ is $\tilde{d}_{\mathfrak{p}'}$-isolated then T is $\mathfrak{p}$-$\aleph_0$-categorical.*

An anomaly with $\mathfrak{p}$-categoricity

In the previous corollary, "T' $\mathfrak{p}$-$\aleph_0$-categorical" would not suffice.

Example.

- $T_0 = \mathrm{Th}(L_p(\mathbb{R}), \wedge, \vee, \ldots)$ theory of L_p Banach lattices [BBH].
- Let $a = \chi_{[0,1]}$, $b = \chi_{[1,2]}$; $T = T_0(a)$, $\mathfrak{p} = \mathrm{id}_T$; $T' = T(b) = T_0(a,b)$, $\mathfrak{p}' = \mathfrak{p}_b$ (So $\mathfrak{p}'$ fixes a and perturbs b).
- T is not $\aleph_0$-categorical. Two models:

$$(L^p[0,1], \chi_{[0,1]}) \quad \not\simeq \quad (L^p[0,2], \chi_{[0,1]}),$$

- But T' is $\mathfrak{p}'$-$\aleph_0$-categorical:

$$(L^p[0,2], \chi_{[0,1]}, \chi_{[0,2]}) \quad \simeq_{\mathfrak{p}'} \quad (L^p[0,3], \chi_{[0,1]}, \chi_{[0,2]}),$$

Perturbations of types with parameters

Say a perturbation system $\mathfrak{p}$ for T is given, and we wish to work over $A \subseteq \bar{M}$. How do we extend $d_{\mathfrak{p}}$ to $\mathrm{S}_n(A)$? We need a perturbation system for $T(A) = \mathrm{Th}_{\mathcal{L}(A)}(\bar{M}, A)$.

- In case $A = \bar{a} = a_{<n}$ is finite, we have already seen one option: use $\mathfrak{p}_{\bar{a}}$.
- In case A is infinite, this doesn't make sense, or else boils down to $\mathfrak{p}/A$ which *fixes* A. For $M, N \vDash T(A)$ (so $A \subseteq M, N$):

$$\theta \in \mathrm{Pert}_{(\mathfrak{p}/A)(r)}(M,N) \Longleftrightarrow \theta \in \mathrm{Pert}_{\mathfrak{p}(r)}(M,N) \& \theta{\restriction}_A = \mathrm{id}_A .$$

$\rightsquigarrow$ T is $\mathfrak{p}$-λ-*stable* if $\|M\| \leq \lambda$ implies $\|(\mathrm{S}_n(M), \tilde{d}_{\mathfrak{p}/M})\| \leq \lambda$.

Expansion by a generic automorphism

Theorem (Chatzidakis, Pillay [CP98]). *Let T be stable, $T_\sigma = T \cup \{$"σ is an automorphism"$\}$. Assuming T_σ has a model-companion T_A:*

1. *T_A is simple.*
2. *If T is superstable, then T_A is supersimple.*

The first part generalises to continuous logic. What about the second part?
Consider $T = PA$ theory of atomless probability algebras. Then:

- PA is superstable (in fact $\aleph_0$-stable).
- (Berenstein, Henson [BH]) PA_A exists and is stable.
- (B.) PA_A is not superstable, and therefore not supersimple.

Let $\mathfrak{p} = \mathrm{id}_T$. $\mathfrak{p}_\sigma$ allows to perturb the automorphism, while fixing the underlying model of T.

Proposition (B., Berenstein). *PA_A is $\mathfrak{p}_\sigma$-superstable, and in fact $\mathfrak{p}_\sigma$-$\aleph_0$-stable.*

Question. Let T be any superstable theory such that T_A exists. Is T_A is $\mathfrak{p}_\sigma$-supersimple?

References

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[BH] Alexander Berenstein and C. Ward Henson, *Model theory of probability spaces with an automorphism*, submitted.

[CP98] Zoé Chatzidakis and Anand Pillay, *Generic structures and simple theories*, Annals of Pure and Applied Logic **95** (1998), 71–92.