Hilbert spaces and their generic automorphisms

Talk at AIM

1. Hilbert spaces

A pre-Hilbert space H over $\mathbb{C}$ ($\mathbb{R}$) is a vector space over $\mathbb{C}$ ($\mathbb{R}$) with an inner product $\langle\ \rangle$ that satifies the following properties:

(1) $\langle ax+by, z\rangle = a\langle x,z\rangle + b\langle y,z\rangle$ for all $x,y,z\in H$, $a,b\in\mathbb{C}$.

(2) $\langle x,y\rangle = \overline{\langle y,x\rangle}$

(3) $\langle x,x\rangle \in (0,\infty)$ for any nonzero $x\in H$.

A pre-Hilbert space which is complete with respect to the norm $\|x\| = \sqrt{\langle x,x\rangle}$ is called a *Hilbert space.*

Let H be an infinite dimensional Hilbert space. We will treat H as a metric structure by writing it in the form

$$\big((B_n(H)\mid n\ge 1), 0, \{I_{mn}\}_{m<n}, \{\lambda\}_{\lambda\in\mathbb{R}}, +, \langle\ \rangle\big)$$

where $B_n(H)=\{x\in H \mid \langle x,x\rangle \le n^2\}$, 0 is the zero vector, $I_{mn}\colon B_m\to B_n$ is the inclusion map for $m<n$, $+\colon B_n(H)\times B_n(H)\to B_{2n}(H)$ is the vector addition, $\lambda_r\colon B_n(H)\to B_{nk}(H)$ where $k-1<|r|<k$ is the scalar multiplication by r. All the operations are uniformly continuous (when restricted to a ball).

The metric on each sort is given by $d(x,y)=\|x-y\|=\sqrt{\langle x-y,x-y\rangle}$ and the diameter of $B_n(H)$ is $2n$ for each $n\ge 1$. Note that the predicate defined by the inner product takes its values in the closed bounded interval $[-n^2,n^2]$ when applied to elements of the sort B_n.

Let $x,y\in H$ and let $A\subset H$. By $\overline{A}$ we mean the normed closure of the linear span of A. We denote by $P_{\bar{A}}(x)$ the projection of x on the subspace $\overline{A}$. We denote by $A^\perp$ the set $\{z\in H : \langle a,z\rangle = 0 \text{ for all } a\in A\}$. The space $A^\perp$ is a closed subspace of H.

1.1. **Lemma.** *Let $c_1,\dots,c_n,d_1,\dots,d_n\in H$ and let $A\subset H$. Then tp$(c_1,\dots,c_n/A)=$ tp$(d_1,\dots,d_n/A)$ if and only if $P_{\bar{A}}(c_i)=P_{\bar{A}}(d_i)$ and $\langle c_i,c_j\rangle=\langle d_i,d_j\rangle$ for $i,j\le n$. Hence $Th(H)$ has quantifier elimination.*

Proof. If $tp(c_1,\dots,c_n/A)=\text{tp}(d_1,\dots,d_n/A)$ then $\langle c_i,c_j\rangle=\langle d_i,d_j\rangle$ for $i,j\le n$ and for every $a,b\in A$, $\langle c_i-b,a\rangle=\langle d_i-b,a\rangle$; thus $P_{\bar{A}}(c_i)=P_{\bar{A}}(d_i)$.

Conversely, assume that $P_{\bar{A}}(c_i) = P_{\bar{A}}(d_i)$ and $\langle c_i, c_j\rangle = \langle d_i, d_j\rangle$ for $i, j \leq n$. Then $c_i - P_{\bar{A}}(c_i), d_i - P_{\bar{A}}(d_i) \in A^{\perp}$ and $\langle c_i - P_{\bar{A}}(c_i), c_j - P_{\bar{A}}(c_j)\rangle = \langle d_i - P_{\bar{A}}(d_i), d_j - P_{\bar{A}}(d_j)\rangle$. After using the Gram-Schmidt orthonormalization process we can build an automorphism of H that fixes $\bar{A}$ pointwise and takes $c_i - P_{\bar{A}}(c_i)$ to $d_i - P_{\bar{A}}(d_i)$ for $i \leq n$. □

Since the theory of Hilbert spaces is separably categorical, then the theory of infinite dimensional Hilbert spaces is complete. Call this theory T_H. Note that to say that a Hilbert space is infinite dimensional we only need to add the scheme (indexed by $n \in \mathbb{N}^+$):

$\inf_{x_1} \dots \inf_{x_n} \max\{|\langle x_i, x_j\rangle - \delta_{ij}| : i, j \leq n\} = 0$

where δ_{ij} equals to 1 if $i = j$ and to zero otherwise.

1.2. **Lemma.** *Let $A \subset H$. Then the definable closure of A equals $\bar{A}$.*

Proof. We show that if $c \in \bar{A}$, then $c \in \mathrm{dcl}(A)$. Given $c \in \bar{A}$, there is a Cauchy sequence $\{c_n : n \geq 1\}$ of elements in the space spanned by A such that $\lim_{n\to\infty} c_n = c$. We may assume that $\|c_n - c\| \leq 1/(2n)$ for $n \geq 1$. Let $\varphi_n(x) = \|x - c_n\| \dot{-} 1/(2n)$. Then the family of formulas $\{\varphi_n(x) \mid n \geq 1\}$ and numbers $\{\delta_n = 1/(2n) \mid n \geq 1\}$ shows that $\{c\}$ is A-definable.

Assume now that $c \notin \bar{A}$, so $c - P_{\bar{A}}(c) \neq 0$. Let $y \in A^{\perp}$ be such that $\|y\| = \|c - P_{\bar{A}}(c)\|$. Then $\mathrm{tp}(c/A) = \mathrm{tp}(P_{\bar{A}}(c) + y/A)$ so $|\{d \mid \mathrm{tp}(d/A) = \mathrm{tp}(c/A)\}|$ is unbounded and thus $c \notin \mathrm{dcl}(A)$. □

1.3. **Proposition.** *Let $x, y \in H$ and let $A \subset H$. Then*

$$d(\mathrm{tp}(x/A), \mathrm{tp}(y/A))^2 = \|P_{\bar{A}}(x) - P_{\bar{A}}(y)\|^2 + \big|\|x - P_{\bar{A}}(x)\| - \|y - P_{\bar{A}}(y)\|\big|^2$$

Proof. Let $x, y \in H$ and let $A \subset H$. If $\mathrm{tp}(x'/A) = \mathrm{tp}(x/A)$ and $\mathrm{tp}(y'/A) = \mathrm{tp}(y/A)$, then

$$\|x' - y'\|^2 = \|P_{\bar{A}}(x') - P_{\bar{A}}(y')\|^2 + \|(x' - P_{\bar{A}}(x')) - (y' - P_{\bar{A}}(y'))\|^2 \geq$$

$$\|P_{\bar{A}}(x) - P_{\bar{A}}(y)\|^2 + \big|\|x - P_{\bar{A}}(x)\| - \|y - P_{\bar{A}}(y)\|\big|^2$$

For the converse, let $x_{\perp} = x - P_{\bar{A}}(x)$ and $y_{\perp} = y - P_{\bar{A}}(y)$. If $x_{\perp} = 0$ the result is clear, so we may assume that $x_{\perp} \neq 0$. Let $\alpha = \|y_{\perp}\|/\|x_{\perp}\|$ and let $z = \alpha x_{\perp}$. Then by Lemma 1.1 $\mathrm{tp}(y/A) = \mathrm{tp}(P_{\bar{A}}(y) + z/A)$ and by the Pythagorean theorem,

$\|x-(P_A(y)+z)\|^2 = \|P_{\bar{A}}(x)-P_{\bar{A}}(y)\|^2 + \left|\|x_\perp - \alpha x_\perp\|\right|^2 = \|P_{\bar{A}}(x)-P_{\bar{A}}(y)\|^2 + \left|\|x_\perp\| - \|y_\perp\|\right|^2$. □

Recall that T_H is separably categorical. Note that for $A=\emptyset$, the formula in the previous proposition gives that for $x,y\in H$, $d(\mathrm{tp}(x),\mathrm{tp}(y)) = |\|x\|-\|y\||$. So for types belonging to the sort $B_n(H)$, we get that $(S_1(T),d)\cong([0,n],l)$ with l the euclidean metric in the interval, which is a compact space.

1.4. **Proposition.** *The theory T_H is ω-stable.*

Proof. Let H be an infinite dimensional Hilbert space, let $A\subset H$ be countable and let $T_A = Th(\mathcal{M},a)_{a\in A}$. We may assume that the dimension of $A^\perp$ (in H) is infinite. Since H is separable, $\bar{A}$ has countable dimension. By lemma 1.1 every 1-type over A is realized in H. Since H is separable with respect to the norm, so is $S_1(A)$ with respect to the d metric. □

A model theoretic study of infinite dimensional Hilbert spaces is carried out in [4, 1]. Most of the material from this section follows those sources. Proposition 1.4 appears in [6].

2. Generic automorphisms

Let U be a new unary function symbol and let L_U be the language of Hilbert spaces expanded with U.

The automorphisms of Hilbert spaces are *unitary maps*. To understand the model Theory of Hilbert spaces with unitary maps we will need some results from spectral theory:

2.1. **Definition.** Let H be a Hilbert space and let U be a unitary map on H. The spectrum of U is the set of all $\lambda\in S^1$ for which there is a sequence $\{v_i : i\in\omega\}$ of normal vectors in H such that $\lim_{i\to\infty}\|U(v_i)-\lambda v_i\| = 0$.

A complex number λ is in the *discrete spectrum* if it is an isolated point in the spectrum of finite multiplicity. That is, $Dim\{x\in H : U(x)=\lambda x\}$ is finite and nonzero. The complement of the discrete spectrum is the *essetial spectrum* and it is denoted by $\sigma_e(U)$.

2.2. **Definition.** Let H be a separable Hilbert space and let U, U' be unitary operators on H. We say that U and U' are *approximately unitarily equivalent* if there is a sequence $\{V_n : n \in \omega\}$ of unitary operators such that $\lim_{n\to\infty} \|U' - V_n U V_n^*\| = 0$.

We also need the following corollary of a Theorem by Weyl-von Neumann-Berg.

2.3. **Theorem.** *Suppose that U, U' are unitary operators on separable Hilbert spaces H, H' respectively. Then U and U' are approximate unitarily equivalent if and only if:*

(1) $\sigma_e(U) = \sigma_e(U')$
(2) $Dim\{x \in H : Ux = \lambda x\} = Dim\{x \in H' : U'x = \lambda x\}$ *for* $\lambda \in S^1 \setminus \sigma_e(U)$

From the previous result we get that the class of pairs (H, U), where H is a Hilbert space and U is a unitary map whose spectrum is the whole set S^1 is an elementary class. Call the theory of this class $T_H A$. From the previous Theorem $T_H A$ is complete.

2.4. **Definition.** Let T be a theory. We say that a metric structure $\mathcal{M} \models T$ is *existentially closed* if whenever $\mathcal{N} \models T$, $\mathcal{N} \supset \mathcal{M}$ and $\mathcal{N} \models \inf_{\bar{x}} \varphi(x, \bar{a}) = r$ where $\bar{a} \in M$, then $\mathcal{M} \models \inf_{\bar{x}} \varphi(x, \bar{a}) = r$.

2.5. **Lemma.** *$T_H A$ is the model companion of $T_H \cup$"U is an automorphism".*

Proof. We only proof that if (H, U) is existentially closed then $(H, U) \models T_H A$. So let (H, U) be existentially closed. Let H' be a separable Hilbert space and let U' be a unitary operator on H' whose spectrum is S^1. Consider the structure $(H \oplus H', U \oplus U')$, where $H \oplus H'$ is the direct sum of the two spaces and $U \oplus U'$ is the induced automorphism of the sum. Then $(H, U) \subset (H \oplus H', U \oplus U')$. Let $\lambda \in S^1$, let $\epsilon > 0$ and let $v' \in H'$ be a normal vector such that $\|U'(v') - \lambda v'\| \leq \epsilon/2$. Since (H, U) is existentially closed, there is $v \in H$ a normal vector such that $\|U(v) - \lambda v\| \leq \epsilon$. Since ϵ was arbitrary, we get that λ belongs to the spectrum of U. Thus the spectrum of U is S^1 and $(H, U) \models T_H A$. □

It is easy to see that $T_H A$ has quantifier elimination and is stable. Stability can be showed by proving that the natural notion of independence satisfies all expected properties. It is easy to show that $T_H A$ is not ω-stable: for each $\lambda \in S^1$ let $p_\lambda(x)$ be the type stating $U(x) = \lambda x$ and $\|x\| = 1$. Then for $\lambda \neq \eta$, $d(p_\lambda, p_\eta) = \sqrt{2}$ and thus $(S_1(T_H A), d)$ is not separable.

2.6. **Proposition.** *$T_H A$ is ω-stable up to perturbations of the automorphism.*

A complete description of $T_H A$ and its properties can be found in [3]. A proof that $T_H A$ is ω-stable up to perturbations of the automorphism can be found in [2]. The Weyl-von Neumann-Berg Theorem can be found in [5].

References

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