Introduction

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Outline

1. Continuous logic
   - Basic definitions
   - Semantics
   - Theories

2. Continuous model theory
   - Types
   - The metric on $S_n(T)$
   - Stability
Many classes of (complete) metric structures arising in analysis are “tame” (e.g., admit well-behaved notions of independence) although not elementary in the classical sense. Continuous logic [BU, BBHU] (Ben-Yaacov, Berenstein, Henson & Usvyatsov) is an attempt to apply model-theoretic tools to such classes. It was preceded by:

- Henson’s logic for Banach structures (positive bounded formulae, approximate satisfaction).
- Ben-Yaacov’s positive logic and compact abstract theories.
- Chang and Keisler’s continuous model theory (1966).
- Łukasiewicz’s many-valued logic (similar, although probably devised for other purposes).
- ...?
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2 Continuous model theory
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   • Stability
Intellectual game: replace $\{T, F\}$ with $[0, 1]$

- The basic idea is: “replace the space of truth values $\{T, F\}$ with $[0, 1]$, and see what happens”…
- Things turn out more elegant if we agree that 0 is “True”.
- Greater truth value is falser.
Ingredient I: non-logical symbols

- A signature $\mathcal{L}$ consists of function and predicate symbols, as usual.
- $n$-ary function symbols: interpreted as functions $M^n \to M$.
- $n$-ary predicate symbols: interpreted as functions $M^n \to [0, 1]$.
- $\mathcal{L}$-terms and atomic $\mathcal{L}$-formulae are as in classical logic.
Probability algebras are Boolean algebras of events in probability spaces; the probability of an event is a value in $[0, 1]$. Language of probability algebras: $\mathcal{L} = \{0, 1, \cdot^c, \cap, \cup, \mu\}$.

- 0, 1 are 0-ary function symbols (constant symbols).
- $\cdot^c$ (complement) is a unary function symbol.
- $\cup, \cap$ (union, intersection) are binary function symbols.
- $\mu$ (probability) is a unary predicate symbol.
Example: language of probability algebras

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Thus:

- $z, x \cap y^c, x \triangle y$ are terms (values in the algebra).
- $\mu(x), \mu(x \cap y^c)$ are atomic formulae (values in $[0,1]$).
Any continuous function \([0, 1]^n \rightarrow [0, 1]\) should be admitted as an \(n\)-ary connective.

Problem: uncountable syntax. But a dense subset of \(C([0, 1]^n, [0, 1])\) (in uniform convergence) is good enough.

The following connectives generate a (countable) dense family of connectives (lattice Stone-Weierstrass):

\[
\neg x := 1 - x; \quad \frac{1}{2} x := x/2; \quad x \div y := \max\{x - y, 0\}.
\]

“\(\varphi \div \psi\)” replaces “\(\psi \rightarrow \varphi\)”. In particular: \(\{\psi, \varphi \div \psi\} \models \varphi\) (Modus Ponens: if \(\psi = 0\) and \(\varphi \div \psi = 0\) then \(\varphi = 0\)).
Ingredient III: Quantifiers

- If $R \subseteq M^{n+1}$ is a predicate on $M$, $(\forall x R(x, \bar{b}))^M$ is the falsest among $\{R(a, \bar{b}) : a \in M\}$.
- By analogy, if $R : M^{n+1} \to [0, 1]$ is a continuous predicate:

$$
(\forall x R(x, \bar{b}))^M = \sup_{a \in M} R(a, \bar{b}).
$$

We will just use “$\sup_x \varphi$” instead of “$\forall x \varphi$”.
- Similarly, “$\exists x \varphi$” becomes “$\inf_x \varphi$”.
- Prenex normal form exists since the connectives $\neg, \frac{1}{2}, \div$ are monotone in each argument:

$$
\varphi \div \inf_x \psi \equiv \sup_x (\varphi \div \psi), \quad \&c. \ldots
$$
We may construct useful connectives:

\[ a \land b := \min\{a, b\} = a - (a \div b) \]
\[ a \lor b := \max\{a, b\} = \neg(\neg a \land \neg b) \]
\[ |a - b| = (a \div b) \lor (b \div a) \]

Thus: \( \mu(x \cap y) \) and \( |\mu(x) - \mu(y)| \) are quantifier-free formulae.

\( \sup_x \inf_y |\mu(x \cap y) - \frac{1}{2}\mu(x)| \) is a formula. It is in fact a sentence, as it has no free variables.
Ingredient IV: Equality...

In classical logic the symbol $=\equiv$ always satisfies:

\[
x = x \quad (x = y) \rightarrow ((x = z) \rightarrow (y = z))
\]  \hspace{1cm} (ER)

Replacing "$x = y$" with "$d(x, y)$" and "$\varphi \rightarrow \psi$" with "$\psi \models \varphi$":

\[
d(x, x) \quad \left( d(y, z) \models d(x, z) \right) \models d(x, y)
\]

I.e., $d$ is a pseudo-metric:

\[
d(x, x) = 0 \quad d(y, z) \leq d(x, z) + d(x, y)
\]  \hspace{1cm} (PM)
Similarly, $=$ is a congruence relation:

$$(x = y) \rightarrow (P(x, \bar{z}) \rightarrow P(y, \bar{z})) \quad \text{(CR)}$$

Translates to:

$$(P(y, \bar{z}) \div P(x, \bar{z}) \div d(x, y))$$

I.e., $P$ is 1-Lipschitz:

$$P(y, \bar{z}) \div P(x, \bar{z}) \leq d(x, y) \quad \text{(1L)}$$
Similarly, $\equiv$ is a congruence relation:

$$(x = y) \rightarrow \left( P(x, \bar{z}) \rightarrow P(y, \bar{z}) \right) \tag{CR}$$

Translates to:

$$\left( P(y, \bar{z}) \equiv P(x, \bar{z}) \right) \equiv d(x, y)$$

I.e., $P$ is 1-Lipschitz:

$$P(y, \bar{z}) \equiv P(x, \bar{z}) \leq d(x, y) \tag{1L}$$

∴ all predicate and function symbols must be 1-Lipschitz in $d$.

**Example (Probability algebras, part 3)**

Indeed: $d(a, b) := \mu(a \triangle b)$ is a (pseudo)metric on events, and each of $\cdot^c, \cup, \cap, \mu$ is 1-Lipschitz.
A set $M$, equipped with a pseudo-metric $d^M$ and 1-Lipschitz interpretations $f^M, P^M$ of symbols $f, P \in \mathcal{L}$ is an $\mathcal{L}$-pre-structure. It is an $\mathcal{L}$-structure if $d^M$ is a complete metric.
A set $M$, equipped with a pseudo-metric $d^M$ and 1-Lipschitz interpretations $f^M, P^M$ of symbols $f, P \in \mathcal{L}$ is an $\mathcal{L}$-pre-structure. It is an $\mathcal{L}$-structure if $d^M$ is a complete metric.

- Once $=^M$ is a congruence relation, classical logic cannot tell whether it is true equality or not.
- Similarly, once all symbols are 1-Lipschitz, continuous logic cannot tell whether:
  - $d^M$ is a true metric or a mere pseudo-metric.
  - A Cauchy sequence has a limit or not.

- A pre-structure $M$ is logically indistinguishable from its completion $\hat{M}/\sim_d$. ($a \sim_d b \iff d(a, b) = 0$)
Recap: probability algebras

- Let \((\Omega, \mathcal{B}, \mu)\) be a probability space.
- Let \(\mathcal{B}_0 \leq \mathcal{B}\) be the null-measure ideal, and \(\mathcal{B} = \mathcal{B}/\mathcal{B}_0\). Then \(\mathcal{B}\) is a Boolean algebra and \(\mu\) induces \(\bar{\mu}: \mathcal{B} \to [0, 1]\). The pair \((\mathcal{B}, \bar{\mu})\) is a probability algebra.
- It admits a complete metric: \(d(a, b) = \bar{\mu}(a \Delta b)\). \(\bar{\mu}\) and the Boolean operations are 1-Lipschitz.
Recap: probability algebras

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  Then \(\mathcal{B}\) is a Boolean algebra and \(\mu\) induces \(\bar{\mu}: \mathcal{B} \to [0, 1]\).
  The pair \((\mathcal{B}, \bar{\mu})\) is a probability algebra.
- It admits a complete metric: \(d(a, b) = \bar{\mu}(a \triangle b)\). \(\bar{\mu}\) and the Boolean operations are 1-Lipschitz.
- \((\mathcal{B}, 0, 1, \cap, \cup, \cdot^c, \mu)\) is a pre-structure;
  \((\mathcal{B}, 0, 1, \cap, \cup, \cdot^c, \bar{\mu})\) is its completion (i.e., a structure).
Let $(\Omega, \mathcal{B}, \mu)$ be a probability space.

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It admits a complete metric: $d(a, b) = \bar{\mu}(a \Delta b)$. $\bar{\mu}$ and the Boolean operations are 1-Lipschitz.

$(\mathcal{B}, 0, 1, \cap, \cup, \cdot^c, \mu)$ is a pre-structure; $(\mathcal{B}, 0, 1, \cap, \cup, \cdot^c, \bar{\mu})$ is its completion (i.e., a structure).

Remark

If $a_n \rightarrow a$ in $d$ then $\mu(a_n) \rightarrow \mu(a)$. Thus $\sigma$-additivity of the measure comes "for free".
By replacing \( \{ T, F \} \) with \([0, 1]\) we obtained a logic for (bounded) complete metric 1-Lipschitz structures.

It is fairly easy to replace “1-Lipschitz” with “uniformly continuous”.

One can also overcome “bounded”, but it’s trickier.

Since all structures are complete metric structures we do not measure their size by cardinality, but by density character:

\[
\| (M, d) \| = \min\{|A| : A \subseteq M \text{ is dense} \}.
\]
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As usual, the notation $\varphi(x_0, \ldots x_{n-1})$ [or $\varphi(x_{<n})$, or $\varphi(\bar{x})$] means that the free variables of $\varphi$ are among $x_0, \ldots, x_{n-1}$.

If $M$ is a structure and $\bar{a} \in M^n$: we define the truth value $\varphi^M(\bar{a}) \in [0, 1]$ inductively, in the “obvious way”.

**Example**

Let $(M, 0, 1, \cdot^c, \cup, \cap, \mu)$ be a probability algebra, $\varphi(x) = \inf_y |\mu(x \cap y) - \frac{1}{2}\mu(x)|$.

- If $a \in M$ is an atom, then $\varphi^M(a) = \frac{1}{2}\mu(a)$.
- If $a$ has no atoms below it then $\varphi^M(a) = 0$. 
As usual, the notation \( \varphi(x_0, \ldots x_{n-1}) \) [or \( \varphi(x_{<n}) \), or \( \varphi(\bar{x}) \)] means that the free variables of \( \varphi \) are among \( x_0, \ldots, x_{n-1} \).

If \( M \) is a structure and \( \bar{a} \in M^n \): we define the truth value \( \varphi^M(\bar{a}) \in [0, 1] \) inductively, in the “obvious way”.

**Example**

Let \((M, 0, 1, \cdot^c, \cup, \cap, \mu)\) be a probability algebra, 
\[
\varphi(x) = \inf_y \left| \mu(x \cap y) - \frac{1}{2} \mu(x) \right|.
\]

- If \( a \in M \) is an atom, then \( \varphi^M(a) = \frac{1}{2} \mu(a) \).
- If \( a \) has no atoms below it then \( \varphi^M(a) = 0 \).

The function \( \varphi^M : M^n \to [0, 1] \) is uniformly continuous (by induction on \( \varphi \)).
Various “elementary” notions

- **Elementary equivalence**: If $M, N$ are two structures then $M \equiv N$ if $\varphi^M = \varphi^N \in [0, 1]$ for every sentence $\varphi$ (i.e.: formula without free variables).

  Equivalently: “$\varphi^M = 0 \iff \varphi^N = 0$ for all sentence $\varphi$.”

- **Elementary extension**: $M \preceq N$ if $M \subseteq N$ and $\varphi^M(\bar{a}) = \varphi^N(\bar{a})$ for every formula $\varphi$ and $\bar{a} \in M$. This implies $M \equiv N$.

**Lemma (Elementary chains)**

The union of an elementary chain $M_0 \preceq M_1 \preceq \ldots$ is an elementary extension of each $M_i$.

Caution: we have to replace the union of a countable increasing chain with its completion.
Ultraproducts

- \( (M_i : i \in I) \) are structures, \( \mathcal{U} \) an ultrafilter on \( I \).
- We let \( N_0 = \prod_{i \in I} M_i \) as a set; its members are \( (\bar{a}) = (a_i : i \in I), a_i \in M_i. \)
- We interpret the symbols:

\[
\begin{align*}
  f^{N_0}((a_i : i \in I), \ldots) &= (f^{M_i}(a_i, \ldots) : i \in I) \quad \in N_0 \\
  P^{N_0}((a_i : i \in I), \ldots) &= \lim_{\mathcal{U}} P^{M_i}(a_i, \ldots) \quad \in [0, 1]
\end{align*}
\]

- This way \( N_0 \) is a pre-structure. We define \( N = \hat{N}_0 \) (the completion), and call it the ultraproduct \( \prod_{i \in I} M_i/\mathcal{U} \).
- The image of \( (\bar{a}) \in N_0 \) in \( N \) is denoted \( (\bar{a})_{\mathcal{U}} \):

\[
(\bar{a})_{\mathcal{U}} = (\bar{b})_{\mathcal{U}} \iff 0 = \lim_{\mathcal{U}} d(a_i, b_i) \quad \left[ = d^{N_0}((\bar{a}), (\bar{b})) \right].
\]
Łoś’s Theorem: for every formula $\varphi(x, y, \ldots)$ and elements $(\bar{a})_U, (\bar{b})_U, \ldots \in N = \prod M_i/U:\n
\varphi^N((\bar{a})_U, (\bar{b})_U, \ldots) = \lim_U \varphi^{M_i}(a_i, b_i, \ldots).

[Easy] $M \equiv N$ ($M$ and $N$ are elementarily equivalent) if and only if $M$ admits an elementary embedding into an ultrapower of $N$.

[Deeper: generalising Keisler & Shelah] $M \equiv N$ if and only if $M$ and $N$ have ultrapowers which are isomorphic.
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A theory $T$ is a set of sentences (closed formulae).

$M \models T \iff \varphi^M = 0$ for all $\varphi \in T$.

We sometimes write $T$ as a set of statements “$\varphi = 0$”. We may also allow as statements things of the form “$\varphi \leq r$”, “$\varphi \geq r$”, “$\varphi = r$”, etc.

If $M$ is any structure then its theory is

$$\text{Th}(M) = \{ \text{“} \varphi = 0 \text{“} : \varphi^M = 0 \} \equiv \{ \text{“} \varphi = r \text{“} : \varphi^M = r \}.$$ 

Theories of this form are called complete (equivalently: complete theories are the maximal satisfiable theories).
Compactness

**Theorem (Compactness)**

A theory is satisfiable if and only if it is finitely satisfiable.

Notice that:

\[ T \equiv \{ \varphi \leq 2^{-n} : n < \omega \& \varphi = 0 \} \in T \].

**Corollary**

Assume that \( T \) is approximately finitely satisfiable. Then \( T \) is satisfiable.
Examples of continuous elementary classes

- Hilbert spaces (infinite dimensional).
- Probability algebras (atomless).
- $L^p$ Banach lattices (atomless).
- Fields with a non-trivial valuation in $\mathbb{R}, +$ (algebraically closed, in characteristic $(p, q)$).

&c. . .

All these examples are complete and admit QE.
Universal theories

- A theory consisting solely of \( \left( \sup_{\bar{x}} \varphi(\bar{x}) \right) = 0 \), where \( \varphi \) is quantifier-free, is called universal. Universal theories are those stable under substructures.
- We may write \( \left( \sup_{\bar{x}} \varphi \right) = 0 \) as \( \forall \bar{x} (\varphi = 0) \).
- Similarly, we may write \( \left( \sup_{\bar{x}} |\varphi - \psi| \right) = 0 \) as \( \forall \bar{x} (\varphi = \psi) \).
- And if \( \sigma, \tau \) are terms: we may write \( \left( \sup_{\bar{x}} d(\sigma, \tau) \right) = 0 \) as \( \forall \bar{x} (\sigma = \tau) \).
The class of probability algebras is axiomatised by:

universal equational axioms of Boolean algebras

\[ \forall xy \; d(x, y) = \mu(x \bigtriangleup y) \]

\[ \forall xy \; \mu(x) + \mu(y) = \mu(x \cap y) + \mu(x \cup y) \]

\[ \mu(1) = 1 \]
The (universal) theory of probability algebras

The class of probability algebras is axiomatised by:

universal equational axioms of Boolean algebras
\[
\forall xy \, d(x, y) = \mu(x \triangle y) \\
\forall xy \, \mu(x) + \mu(y) = \mu(x \cap y) + \mu(x \cup y) \\
\mu(1) = 1
\]

The model completion is the \( \forall \exists \)-theory of atomless probability algebras:

\[
\sup_x \inf_y |\mu(x \cap y) - \frac{1}{2}\mu(x)| = 0.
\]
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Types (without parameters)

**Definition**

Let $M$ be a structure, $\bar{a} \in M^n$. Then:

$$tp^M(\bar{a}) = \{ \varphi(\bar{x}) = r : \varphi(\bar{x}) \in \mathcal{L}, r = \varphi(\bar{a})^M \}.$$

$S_n(T)$ is the space of types of $n$-tuples in models of $T$. If $p \in S_n(T)$:

$$\varphi(\bar{x})^p = r \iff \varphi(\bar{x}) = r \in p.$$
Types (without parameters)

Definition

Let $M$ be a structure, $\bar{a} \in M^n$. Then:

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$S_n(T)$ is the space of types of $n$-tuples in models of $T$. If $p \in S_n(T)$:

$$\varphi(\bar{x})^p = r \iff "\varphi(\bar{x}) = r" \in p.$$ 

- The logic topology on $S_n(T)$ is minimal such that $p \mapsto \varphi^p$ is continuous for all $\varphi$.
- This is the analogue of the Stone topology in classical logic; it is compact and Hausdorff (not totally disconnected).
Types (with parameters)

**Definition**

Let $M$ be a structure, $\bar{a} \in M^n$, $B \subseteq M$. Then:

$$tp^M(\bar{a}/B) = \{ \"\varphi(\bar{x}, \bar{b}) = r\" : \varphi(\bar{x}, \bar{y}) \in \mathcal{L}, \bar{b} \in B^m, r = \varphi(\bar{a}, \bar{b})^M \}.$$ 

$S_n(B)$ is the space of types over $B$ of $n$-tuples in **elementary extensions** of $M$. If $p \in S_n(B)$, $\bar{b} \in B$:

$$\varphi(\bar{x}, \bar{b})^p = r \iff \"\varphi(\bar{x}, \bar{b}) = r\" \in p.$$ 

The **logic topology** on $S_n(B)$ is minimal such that $p \mapsto \varphi(\bar{x}, \bar{b})^p$ is continuous for all $\varphi(\bar{x}, \bar{b}), \bar{b} \in B^m$. It is compact and Hausdorff.
Saturated and homogeneous models

**Definition**

Let $\kappa$ be a cardinal, $M$ a structure.

- $M$ is $\kappa$-saturated if for every $A \subseteq M$ such that $|A| < \kappa$ and every $p \in S_1(A)$: $p$ is realised in $M$.

- $M$ is $\kappa$-homogeneous if for every $A \subseteq M$ such that $|A| < \kappa$ and every mapping $f : A \to M$ which preserves truth values, $f$ extends to an automorphism of $M$.

**Fact**

Let $M$ be any structure and $\mathcal{U}$ a non-principal ultrafilter on $\aleph_0$. Then the ultrapower $M^{\aleph_0}/\mathcal{U}$ is $\aleph_1$-saturated.
A **monster model** of a complete theory $T$ is a model of $T$ which is $\kappa$-saturated and $\kappa$-homogeneous for some $\kappa$ which is much larger than any set under consideration.

**Fact**

- Every complete theory $T$ has a monster model.
- If $\bar{M}$ is a monster model for $T$, then every “small” model of $T$ (i.e., smaller than $\kappa$) is isomorphic to some $N \leq \bar{M}$.
- If $A \subseteq \bar{M}$ is small then $S_n(A)$ is the set of orbits in $\bar{M}^n$ under $\mathrm{Aut}(\bar{M}/A)$.

Thus monster models serve as “universal domains”: everything happens inside, and the automorphism group is large enough.
Definable predicates

- We identify a formula $\varphi(x < n)$ with the function $\varphi: S_n(T) \to [0,1]$ it induces: $p \mapsto \varphi^p$. By Stone-Weierstrass these functions are dense in $C(S_n(T), [0,1])$.

- An arbitrary continuous function $\psi: S_n(T) \to [0,1]$ is called a definable predicate. It is a uniform limit of formulae: $\psi = \lim_{n \to \infty} \varphi_n$. Its interpretation:

$$\psi^M(\bar{a}) = \lim_n \varphi_n^M(\bar{a}).$$

Since each $\varphi_n^M$ is uniformly continuous, so is $\psi^M$. 

Definable predicates

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- An arbitrary continuous function $\psi : S_n(T) \to [0, 1]$ is called a **definable predicate**. It is a uniform limit of formulae: $\psi = \lim_{n \to \infty} \varphi_n$. Its interpretation:

  $$\psi^M(\bar{a}) = \lim_n \varphi^M_n(\bar{a}).$$

  Since each $\varphi^M_n$ is uniformly continuous, so is $\psi^M$.

- Same applies with parameters. Note that a definable predicate $\lim \varphi_n(\bar{x}, \bar{b}_n)$ may depend on countably many parameters.
In continuous logic **imaginary elements** are introduced as canonical parameters of formulae and predicates with parameters. Imaginary sorts are also metric:

\[ d(cp(\psi), cp(\chi)) = \sup_{\bar{x}} |\psi(\bar{x}) - \chi(\bar{x})|. \]

- An element \( a \) is **algebraic** over \( A \) if the set of its conjugates over \( A \) is compact (replaces “finite”).
- \( acl^{eq}(A) \) is the set of all imaginaries algebraic over \( A \).
Theorem (Omitting types)

Assume $T$ is countable and $X \subseteq S_1(T)$ is meagre (i.e., contained in a countable union of closed nowhere-dense sets). Then $T$ has a model $M$ such that a dense subset of $M$ omits each type in $X$. (Similarly with $X_n \subseteq S_n(T)$ meagre for each $n$.)
Theorem (Omitting types)

Assume $T$ is countable and $X \subseteq S_1(T)$ is meagre (i.e., contained in a countable union of closed nowhere-dense sets). Then $T$ has a model $M$ such that a dense subset of $M$ omits each type in $X$. (Similarly with $X_n \subseteq S_n(T)$ meagre for each $n$.)

What about omitting types in $M$, and not only in a dense subset?
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The topological structure of $S_n(T)$ is insufficient. We will also need to consider the distance between types:

$$d(p, q) = \inf \{d(a, b) : a, b \in M \models T \land M \models p(a) \cup q(b)\}.$$ 

(In case $T$ is incomplete and $p, q$ belong to different completions: $d(p, q) = \inf \emptyset := \infty$.)
Metric on types

The topological structure of $S_n(T)$ is insufficient. We will also need to consider the distance between types:

$$d(p, q) = \inf \{d(a, b) : a, b \in M \models T \& M \models p(a) \cup q(b)\}.$$  

(In case $T$ is incomplete and $p, q$ belong to different completions: $d(p, q) = \inf \emptyset := \infty$.)

The infimum is always attained as minimum. Indeed, apply compactness to the partial type:

$$p(x) \cup q(y) \cup \{d(x, y) \leq d(p, q) + 2^{-n} : n < \omega\}.$$
Some properties of \((S_n(T), d)\)

- If \(f : S_n(T) \to [0, 1]\) is topologically continuous (\(f\) is a definable predicate) then it is metrically uniformly continuous.
- Implies: The metric refines the topology.
- If \(F \subseteq S_n(T)\) is closed, then so is the set:
  \[
  \bar{B}(F, r) = \{ p \in S_n(T) : d(p, F) \leq r \}.
  \]
- Implies: \((S_n(T), d)\) is complete.
- And: If \(F \subseteq S_n(T)\) is closed and \(p \in S_n(T)\), then there is \(q \in F\) such that \(d(p, q) = d(p, F)\).
All these properties are consequences of compactness + “metric Hausdorff” property:

**Lemma**

The distance function $d : S_n(T)^2 \rightarrow [0, \infty]$ is lower semi-continuous. That is to say that $\{(p, q) : d(p, q) \leq r\}$ is closed for all $r$.

**Proof.**

The projection $S_{2n}(T) \rightarrow S_n(T) \times S_n(T)$ is closed, and $[d(\bar{x}, \bar{y}) \leq r] \subseteq S_{2n}(T)$ is closed, whereby so is its image $\{(p, q) : d(p, q) \leq r\} \subseteq S_n(T)^2$. □
### Definition

A type $p \in S_n(T)$ is **$d$-isolated** if for all $r > 0$ the metric ball $B(p, r)$ contains $p$ in its topological interior: $p \in B(p, r)^\circ$ (i.e., the metric and the topology coincide at $p$).

### Fact

A type $p \in S_n(T)$ is $d$-isolated if and only if it is **weakly $d$-isolated**, i.e., iff for all $r > 0$: $B(x, r)^\circ \neq \emptyset$. 
Omitting and realising types in models

**Proposition (Henson)**

A d-isolated type $p$ is realised in every model of $T$. If $T$ is countable, then the converse is also true.

**Proof.**

$\implies$ As $B(p, 2^{-n})^\circ \neq \varnothing$ for all $n$, it must be realised in $M$, say by $a_n$. We can furthermore arrange that $d(a_n, a_{n+1}) < 2^{-n-1}$. Then $a_n \rightarrow a \models p$.

$\impliedby$ If $\bar{B}(p, r)^\circ = \varnothing$ for some $r > 0$, we can omit it in a dense subset of $M$. Then $M$ omits $p$.  


Ryll-Nardzewski Theorem

**Definition**

A theory $T$ is $\lambda$-categorical if for all $M, N \models T$:

$$\|M\| = \|N\| = \lambda \implies M \simeq N.$$ 

**Theorem (Henson)**

For a complete countable theory $T$, TFAE:

- $T$ is $\aleph_0$-categorical (unique separable model).
- Every $n$-type over $\emptyset$ is $d$-isolated for all $n$.
- The metric and topology coincide on each $(S_n(T), d)$.
- Every automorphism-invariant uniformly continuous predicate on $\bar{M}$ is definable.
Outline

1. Continuous logic
   - Basic definitions
   - Semantics
   - Theories

2. Continuous model theory
   - Types
   - The metric on $S_n(T)$
   - Stability
Stable theories

Recall: $\| \cdot \|$ denotes the metric density character.

**Definition**

- (Iovino) We say that $T$ is $\lambda$-stable if $\|A\| \leq \lambda \implies \|S_n(A)\| \leq \lambda$.
- It is **stable** if it is $\lambda$-stable for some $\lambda$.
- It is **superstable** if it is $\lambda$-stable for all $\lambda$ big enough.

**Proposition**

The following are equivalent:

- $T$ is stable.
- If $\|M\| \leq 2^{|T|}$ then $|S_n(M)| \leq 2^{|T|}$. 

AIM Workshop: Model theory of metric structures
Notions of independence

Let $\bar{M}$ be a monster model, and $\downarrow_B$ a ternary notion of independence between small subsets of $\bar{M}$: $A \downarrow_B C$ means “$A$ is independence from $C$ over $B$.” It may satisfy:

- **Invariance:** under automorphisms of $\bar{M}$.
- **Symmetry:** $A \downarrow_B C \iff C \downarrow_B A$.
- **Transitivity:** $A \downarrow_B CD \iff \left[ A \downarrow_B C \text{ and } A \downarrow_{BC} D \right]$.
- **Finite character:** $A \downarrow_B C \iff \bar{a} \downarrow_B C$ for all finite $\bar{a} \in A$. 


Notions of independence

Let \( \tilde{M} \) be a monster model, and \( \Downarrow \) a ternary notion of independence between small subsets of \( \tilde{M} \): \( A \Downarrow_B C \) means “\( A \) is independence from \( C \) over \( B \).” It may satisfy:

- **Invariance**: under automorphisms of \( \tilde{M} \).
- **Symmetry**: \( A \Downarrow_B C \iff C \Downarrow_B A \).
- **Transitivity**: \( A \Downarrow_B CD \iff [A \Downarrow_B C \text{ and } A \Downarrow_{BC} D] \).
- **Finite character**: \( A \Downarrow_B C \iff \bar{a} \Downarrow_B C \) for all finite \( \bar{a} \in A \).
- **Extension**: for all \( \bar{a}, B, C \) there is \( \bar{a}' \) such that
  \[ \text{tp}(\bar{a}/B) = \text{tp}(\bar{a}'/B) \text{ and } \bar{a}' \Downarrow_B C. \]
- **Local character**: For all \( \bar{a} \) and \( B \) there is \( B_0 \subseteq B \) such that
  \[ |B_0| \leq |T| \text{ and } \bar{a} \Downarrow_{B_0} B. \]
- **Stationarity**: if \( M \preceq \tilde{M}, \bar{a} \Downarrow_M B, \bar{a}' \Downarrow_M B \) then:
  \[ \text{tp}(\bar{a}/M) = \text{tp}(\bar{a}'/M) \implies \text{tp}(\bar{a}/B) = \text{tp}(\bar{a}'/B). \]
Stability and independence

Theorem

$T$ is stable if and only if its monster models admit notions of independence satisfying all of the above. Moreover, if such a notion of independence exists then it is unique.

Example

- In Hilbert spaces: $\perp$ = orthogonality.
- In probability algebras: $\perp$ = probabilistic independence.
- In $L^p$ lattices: more complicated.