Introduction

Itaï Ben-Yaacov
C. Ward Henson

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Outline

1. Continuous logic
   - Basic definitions
   - Semantics
   - Theories
2. Continuous model theory
   - Types
   - The metric on $S_n(T)$
   - Stability

Origins

Many classes of (complete) metric structures arising in analysis are "tame" (e.g., admit well-behaved notions of independence) although not elementary in the classical sense.

Continuous logic [BU, BBHU] (Ben-Yaacov, Berenstein, Henson & Usvyatsov) is an attempt to apply model-theoretic tools to such classes. It was preceded by:

- Henson's logic for Banach structures (positive bounded formulae, approximate satisfaction).
- Ben-Yaacov's positive logic and compact abstract theories.
- Chang and Keisler's continuous model theory (1966).
- Łukasiewicz's many-valued logic (similar, although probably devised for other purposes).
- ...?

Outline

Intellectual game: replace $\{T, F\}$ with $[0, 1]$

- The basic idea is: "replace the space of truth values $\{T, F\}$ with $[0, 1]$, and see what happens"...
- Things turn out more elegant if we agree that 0 is *"True"*.
- Greater truth value is falser.

Ingredient I: non-logical symbols

- A signature $\mathcal{L}$ consists of function and predicate symbols, as usual.
- n-ary function symbols: interpreted as functions $M^n \to M$.
- n-ary predicate symbols: interpreted as functions $M^n \to [0,1]$.
- $\mathcal{L}$-terms and atomic $\mathcal{L}$-formulae are as in classical logic.

Example: language of probability algebras

Probability algebras are Boolean algebras of events in probability spaces; the probability of an event is a value in $[0,1]$.
Language of probability algebras: $\mathcal{L} = \{0, 1, \cdot^c, \cap, \cup, \mu\}$.

- 0, 1 are 0-ary function symbols (constant symbols).
- $\cdot^c$ (complement) is a unary function symbol.
- $\cup$, $\cap$ (union, intersection) are binary function symbols.
- μ (probability) is a unary predicate symbol.

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Thus:

- $z, x \cap y^c, x \triangle y$ are terms (values in the algebra).
- $\mu(x), \mu(x \cap y^c)$ are atomic formulae (values in $[0,1]$).

Ingredient II: Connectives

- Any continuous function $[0,1]^n \to [0,1]$ should be admitted as an n-ary connective.
- Problem: uncountable syntax. But a dense subset of $C([0,1]^n,[0,1])$ (in uniform convergence) is good enough.
- The following connectives generate a (countable) dense family of connectives (lattice Stone-Weierstrass):

$$\neg x := 1 - x; \quad \tfrac{1}{2}x := x/2; \quad x \dot{-} y := \max\{x - y, 0\}.$$

- "$\varphi \dot{-} \psi$" replaces "$\psi \to \varphi$". In particular: $\{\psi, \varphi \dot{-} \psi\} \vDash \varphi$ (Modus Ponens: if $\psi = 0$ and $\varphi \dot{-} \psi = 0$ then $\varphi = 0$).

Ingredient III: Quantifiers

- If $R \subseteq M^{n+1}$ is a predicate on M, $\left(\forall x R(x, \bar{b})\right)^M$ is the falsest among $\{R(a, \bar{b}) \colon a \in M\}$.
- By analogy, if $R \colon M^{n+1} \to [0,1]$ is a continuous predicate:

$$\left(\forall x\, R(x, \bar{b})\right)^M = \sup_{a \in M} R(a, \bar{b}).$$

  We will just use "$\sup_x \varphi$" instead of "$\forall x \varphi$".
- Similarly, "$\exists x \varphi$" becomes "$\inf_x \varphi$".
- Prenex normal form exists since the connectives $\neg, \frac{1}{2}, \dot{-}$ are monotone in each argument:

$$\varphi \dot{-} \inf_x \psi \equiv \sup_x (\varphi \dot{-} \psi), \qquad \&c\ldots$$

Probability algebras, cntd.

- We may construct useful connectives:

$$
\begin{aligned}
a \wedge b &:= \min\{a, b\} &&= a \dot{-} (a \dot{-} b)\\
a \vee b &:= \max\{a, b\} &&= \neg(\neg a \wedge \neg b)\\
|a - b| & &&= (a \dot{-} b) \vee (b \dot{-} a)
\end{aligned}
$$

- Thus: $\mu(x \cap y)$ and $|\mu(x) - \mu(y)|$ are quantifier-free formulae.
- $\sup_x \inf_y \left|\mu(x \cap y) - \frac{1}{2}\mu(x)\right|$ is a formula. It is in fact a sentence, as it has no free variables.

Ingredient IV: Equality...?

In classical logic the symbol $=$ always satisfies:

$$x = x \qquad (x = y) \to \big((x = z) \to (y = z)\big) \tag{ER}$$

Replacing "$x = y$" with "$d(x, y)$" and "$\varphi \to \psi$" with "$\psi \dot{-} \varphi$":

$$d(x, x) \qquad \big(d(y, z) \dot{-} d(x, z)\big) \dot{-} d(x, y)$$

I.e., d is a pseudo-metric:

$$d(x, x) = 0 \qquad d(y, z) \leq d(x, z) + d(x, y) \tag{PM}$$

Similarly, $=$ is a congruence relation:

$$(x = y) \to \Big(P(x, \bar{z}) \to P(y, \bar{z})\Big) \tag{CR}$$

Translates to:

$$\Big(P(y, \bar{z}) \dot{-} P(x, \bar{z})\Big) \dot{-} d(x, y)$$

I.e., P is 1-Lipschitz:

$$P(y, \bar{z}) \dot{-} P(x, \bar{z}) \leq d(x, y) \tag{1L}$$

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$\therefore$ all predicate and function symbols must be 1-Lipschitz in d.

Example (Probability algebras, part 3)

Indeed: $d(a, b) := \mu(a \triangle b)$ is a (pseudo)metric on events, and each of $\cdot^c, \cup, \cap, \mu$ is 1-Lipschitz.

Structures

Definition

A set M, equipped with a pseudo-metric d^M and 1-Lipschitz interpretations f^M, P^M of symbols $f, P \in \mathcal{L}$ is an $\mathcal{L}$-pre-structure. It is an $\mathcal{L}$-structure if d^M is a complete metric.

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- Once $=^M$ is a congruence relation, classical logic cannot tell whether it is true equality or not.
- Similarly, once all symbols are 1-Lipschitz, continuous logic cannot tell whether:
  - d^M is a true metric or a mere pseudo-metric.
  - A Cauchy sequence has a limit or not.
- A pre-structure M is logically indistinguishable from its completion $\widehat{M/{\sim_d}}$. $(a \sim_d b \Longleftrightarrow d(a, b) = 0)$

Recap: probability algebras

- Let $(\Omega, \mathfrak{B}, \mu)$ be a probability space.
- Let $\mathfrak{B}_0 \leq \mathfrak{B}$ be the null-measure ideal, and $\bar{\mathfrak{B}} = \mathfrak{B}/\mathfrak{B}_0$. Then $\bar{\mathfrak{B}}$ is a Boolean algebra and μ induces $\bar{\mu} \colon \bar{\mathfrak{B}} \to [0, 1]$. The pair $(\bar{\mathfrak{B}}, \bar{\mu})$ is a probability algebra.
- It admits a complete metric: $d(a, b) = \bar{\mu}(a \triangle b)$. $\bar{\mu}$ and the Boolean operations are 1-Lipschitz.

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- It admits a complete metric: $d(a, b) = \bar{\mu}(a \triangle b)$. $\bar{\mu}$ and the Boolean operations are 1-Lipschitz.
- $(\mathfrak{B}, 0, 1, \cap, \cup, \cdot^c, \mu)$ is a pre-structure; $(\bar{\mathfrak{B}}, 0, 1, \cap, \cup, \cdot^c, \bar{\mu})$ is its completion (i.e., a structure).

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Remark

If $a_n \to a$ in d then $\mu(a_n) \to \mu(a)$. Thus σ-additivity of the measure comes "for free".

Bottom line

- By replacing $\{T, F\}$ with $[0, 1]$ we obtained a logic for (bounded) complete metric 1-Lipschitz structures.
- It is fairly easy to replace "1-Lipschitz" with "uniformly continuous".
- One can also overcome "bounded", but it's trickier.
- Since all structures are complete metric structures we do not measure their size by cardinality, but by density character:

$$\|(M, d)\| = \min\{|A| : A \subseteq M \text{ is dense}\}.$$

Outline

Semantics

As usual, the notation $\varphi(x_0, \ldots x_{n-1})$ [or $\varphi(x_{<n})$, or $\varphi(\bar{x})$] means that the free variables of φ are among $x_0, \ldots, x_{n-1}$.

If M is a structure and $\bar{a} \in M^n$: we define the truth value $\varphi^M(\bar{a}) \in [0,1]$ inductively, in the "obvious way".

Example

Let $(M, 0, 1, .^c, \cup, \cap, \mu)$ be a probability algebra, $\varphi(x) = \inf_y \left|\mu(x \cap y) - \frac{1}{2}\mu(x)\right|$.

- If $a \in M$ is an atom, then $\varphi^M(a) = \frac{1}{2}\mu(a)$.
- If a has no atoms below it then $\varphi^M(a) = 0$.

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The function $\varphi^M \colon M^n \to [0,1]$ is uniformly continuous (by induction on φ).

Various "elementary" notions

- Elementary equivalence: If M, N are two structures then $M \equiv N$ if $\varphi^M = \varphi^N \in [0,1]$ for every sentence φ (i.e.: formula without free variables).
  Equivalently: "$\varphi^M = 0 \iff \varphi^N = 0$ for all sentence φ."
- Elementary extension: $M \preceq N$ if $M \subseteq N$ and $\varphi^M(\bar{a}) = \varphi^N(\bar{a})$ for every formula φ and $\bar{a} \in M$. This implies $M \equiv N$.

Lemma (Elementary chains)

The union of an elementary chain $M_0 \preceq M_1 \preceq \ldots$ is an elementary extension of each M_i.

Caution: we have to replace the union of a countable increasing chain with its completion.

Ultraproducts

- $(M_i \colon i \in I)$ are structures, $\mathscr{U}$ an ultrafilter on I.
- We let $N_0 = \prod_{i \in I} M_i$ as a set; its members are $(\bar{a}) = (a_i \colon i \in I)$, $a_i \in M_i$.
- We interpret the symbols:

$$\begin{aligned} f^{N_0}\big((a_i \colon i \in I), \ldots\big) &= \big(f^{M_i}(a_i, \ldots) \colon i \in I\big) && \in N_0 \\ P^{N_0}\big((a_i \colon i \in I), \ldots\big) &= \lim_{\mathscr{U}} P^{M_i}(a_i, \ldots) && \in [0,1] \end{aligned}$$

- This way N_0 is a pre-structure. We define $N = \widehat{N_0}$ (the completion), and call it the ultraproduct $\prod_{i \in I} M_i / \mathscr{U}$.
- The image of $(\bar{a}) \in N_0$ in N is denoted $(\bar{a})_{\mathscr{U}}$:

$$(\bar{a})_{\mathscr{U}} = (\bar{b})_{\mathscr{U}} \iff 0 = \lim_{\mathscr{U}} d(a_i, b_i) \quad \Big[= d^{N_0}\big((\bar{a}), (\bar{b})\big) \Big].$$

Properties of ultraproducts

- Łoś's Theorem: for every formula $\varphi(x, y, \ldots)$ and elements $(\bar{a})_{\mathscr{U}}, (\bar{b})_{\mathscr{U}}, \ldots \in N = \prod M_i/\mathscr{U}$:

$$\varphi^N\big((\bar{a})_{\mathscr{U}}, (\bar{b})_{\mathscr{U}}, \ldots\big) = \lim_{\mathscr{U}} \varphi^{M_i}(a_i, b_i, \ldots).$$

- [Easy] $M \equiv N$ (M and N are elementarily equivalent) if and only if M admits an elementary embedding into an ultrapower of N.
- [Deeper: generalising Keisler & Shelah] $M \equiv N$ if and only if M and N have ultrapowers which are isomorphic.

Outline

Theories

- A theory T is a set of sentences (closed formulae).
- $M \vDash T \iff \varphi^M = 0$ for all $\varphi \in T$.
- We sometimes write T as a set of statements "$\varphi = 0$". We may also allow as statements things of the form "$\varphi \leq r$", "$\varphi \geq r$", "$\varphi = r$", etc.
- If M is any structure then its theory is

$$\mathrm{Th}(M) = \{\text{“}\varphi = 0\text{”} : \varphi^M = 0\} \quad \left[\equiv \{\text{“}\varphi = r\text{”} : \varphi^M = r\}\right].$$

  Theories of this form are called complete (equivalently: complete theories are the maximal satisfiable theories).

Compactness

Theorem (Compactness)

A theory is satisfiable if and only if it is finitely satisfiable.

Notice that:

$$T \equiv \{\text{“}\varphi \leq 2^{-n}\text{”} : n < \omega \,\&\, \text{“}\varphi = 0\text{”} \in T\}.$$

Corollary

Assume that T is approximately finitely satisfiable. Then T is satisfiable.

Examples of continuous elementary classes

- Hilbert spaces (infinite dimensional).
- Probability algebras (atomless).
- L^p Banach lattices (atomless).
- Fields with a non-trivial valuation in $(\mathbb{R}, +)$ (algebraically closed, in characteristic (p, q)).
- &c...

All these examples are complete and admit QE.

Universal theories

- A theory consisting solely of "$\left(\sup_{\bar{x}} \varphi(\bar{x})\right) = 0$", where φ is quantifier-free, is called universal. Universal theories are those stable under substructures.
- We may write $\left(\sup_{\bar{x}} \varphi\right) = 0$ as $\forall \bar{x} \left(\varphi = 0\right)$.
- Similarly, we may write $\left(\sup_{\bar{x}} |\varphi - \psi|\right) = 0$ as $\forall \bar{x} \left(\varphi = \psi\right)$.
- And if σ, τ are terms: we may write $\left(\sup_{\bar{x}} d(\sigma, \tau)\right) = 0$ as $\forall \bar{x} (\sigma = \tau)$.

The (universal) theory of probability algebras

The class of probability algebras is axiomatised by:

universal equational axioms of Boolean algebras

$$\forall xy\, d(x,y) = \mu(x \triangle y)$$

$$\forall xy\, \mu(x) + \mu(y) = \mu(x \cap y) + \mu(x \cup y)$$

$$\mu(1) = 1$$

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The model completion is the $\forall\exists$-theory of atomless probability algebras:

$$\sup_x \inf_y \left|\mu(x \cap y) - \tfrac{1}{2}\mu(x)\right| = 0.$$

Outline

Types (without parameters)

Definition

Let M be a structure, $\bar{a} \in M^n$. Then:

$$\mathrm{tp}^M(\bar{a}) = \{\text{“}\varphi(\bar{x}) = r\text{”} : \varphi(\bar{x}) \in \mathcal{L}, r = \varphi(\bar{a})^M\}.$$

$S_n(T)$ is the space of types of n-tuples in models of T. If $p \in S_n(T)$:

$$\varphi(\bar{x})^p = r \iff \text{“}\varphi(\bar{x}) = r\text{”} \in p.$$

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$\mathrm{S}_n(T)$ is the space of types of n-tuples in models of T. If $p \in \mathrm{S}_n(T)$:

$$\varphi(\bar{x})^p = r \quad \Longleftrightarrow \quad \text{“}\varphi(\bar{x}) = r\text{”} \in p.$$

- The logic topology on $S_n(T)$ is minimal such that $p \mapsto \varphi^p$ is continuous for all φ.
- This is the analogue of the Stone topology in classical logic; it is compact and Hausdorff (not totally disconnected).

Types (with parameters)

Definition

Let M be a structure, $\bar{a} \in M^n$, $B \subseteq M$. Then:

$$\mathrm{tp}^M(\bar{a}/B) = \{\text{“}\varphi(\bar{x}, \bar{b}) = r\text{”} : \varphi(\bar{x}, \bar{y}) \in \mathcal{L}, \bar{b} \in B^m, r = \varphi(\bar{a}, \bar{b})^M\}.$$

$S_n(B)$ is the space of types over B of n-tuples in elementary extensions of M. If $p \in S_n(B)$, $\bar{b} \in B$:

$$\varphi(\bar{x}, \bar{b})^p = r \iff \text{“}\varphi(\bar{x}, \bar{b}) = r\text{”} \in p.$$

The logic topology on $S_n(B)$ is minimal such that $p \mapsto \varphi(\bar{x}, \bar{b})^p$ is continuous for all $\varphi(\bar{x}, \bar{b})$, $\bar{b} \in B^m$. It is compact and Hausdorff.

Saturated and homogeneous models

Definition

Let κ be a cardinal, M a structure.

- M is κ-saturated if for every $A \subseteq M$ such that $|A| < \kappa$ and every $p \in S_1(A)$: p is realised in M.
- M is κ-homogeneous if for every $A \subseteq M$ such that $|A| < \kappa$ and every mapping $f : A \to M$ which preserves truth values, f extends to an automorphism of M.

Fact

Let M be any structure and $\mathscr{U}$ a non-principal ultrafilter on $\aleph_0$. Then the ultrapower $M^{\aleph_0}/\mathscr{U}$ is $\aleph_1$-saturated.

Monster models

A monster model of a complete theory T is a model of T which is κ-saturated and κ-homogeneous for some κ which is much larger than any set under consideration.

Fact

- *Every complete theory T has a monster model.*
- *If $\bar{M}$ is a monster model for T, then every "small" model of T (i.e., smaller than κ) is isomorphic to some $N \preceq \bar{M}$.*
- *If $A \subseteq \bar{M}$ is small then $S_n(A)$ is the set of orbits in $\bar{M}^n$ under* $\mathrm{Aut}(\bar{M}/A)$.

Thus monster models serve as "universal domains": everything happens inside, and the automorphism group is large enough.

Definable predicates

- We identify a formula $\varphi(x_{<n})$ with the function $\varphi\colon S_n(T) \to [0,1]$ it induces: $p \mapsto \varphi^p$. By Stone-Weierstrass these functions are dense in $C(S_n(T),[0,1])$.
- An arbitrary continuous function $\psi\colon S_n(T) \to [0,1]$ is called a definable predicate. It is a uniform limit of formulae: $\psi = \lim_{n\to\infty} \varphi_n$. Its interpretation:

$$\psi^M(\bar{a}) = \lim_n \varphi_n^M(\bar{a}).$$

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  Since each φ_n^M is uniformly continuous, so is ψ^M.
- Same applies with parameters. Note that a definable predicate $\lim \varphi_n(\bar{x}, \bar{b}_n)$ may depend on countably many parameters.

Imaginaries and algebraic closure

- In continuous logic imaginary elements are introduced as canonical parameters of formulae and predicates with parameters. Imaginary sorts are also metric:

$$d(\mathrm{cp}(\psi), \mathrm{cp}(\chi)) = \sup_{\bar{x}} |\psi(\bar{x}) - \chi(\bar{x})|.$$

- An element a is algebraic over A if the set of its conjugates over A is compact (replaces "finite").
- $\mathrm{acl}^{eq}(A)$ is the set of all imaginaries algebraic over A.

Omitting types

Theorem (Omitting types)

Assume T is countable and $X \subseteq S_1(T)$ is meagre (i.e., contained in a countable union of closed nowhere-dense sets). Then T has a model M such that a dense subset of M omits each type in X. (Similarly with $X_n \subseteq S_n(T)$ meagre for each n.)

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What about omitting types in M, and not only in a dense subset?

Outline

Metric on types

The topological structure of $S_n(T)$ is insufficient. We will also need to consider the distance between types:

$$d(p,q) = \inf\{d(a,b)\colon a,b \in M \vDash T \,\&\, M \vDash p(a) \cup q(b)\}.$$

(In case T is incomplete and p, q belong to different completions: $d(p,q) = \inf \varnothing := \infty$.)

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(In case T is incomplete and p,q belong to different completions: $d(p,q) = \inf \varnothing := \infty$.)

The infimum is always attained as minimum. Indeed, apply compactness to the partial type:

$$p(x) \cup q(y) \cup \{d(x,y) \leq d(p,q) + 2^{-n} \colon n < \omega\}.$$

Some properties of $(S_n(T), d)$

- If $f\colon S_n(T) \to [0,1]$ is topologically continuous (f is a definable predicate) then it is metrically uniformly continuous.
- Implies: The metric refines the topology.
- If $F \subseteq S_n(T)$ is closed, then so is the set:

$$\bar{B}(F, r) = \{p \in S_n(T)\colon d(p, F) \leq r\}.$$

- Implies: $(S_n(T), d)$ is complete.
- And: If $F \subseteq S_n(T)$ is closed and $p \in S_n(T)$, then there is $q \in F$ such that $d(p, q) = d(p, F)$.

All these properties are consequences of compactness + "metric Hausdorff" property:

Lemma

The distance function $d\colon S_n(T)^2 \to [0,\infty]$ is lower semi-continuous. That is to say that $\{(p,q)\colon d(p,q) \le r\}$ is closed for all r.

Proof.

The projection $S_{2n}(T) \to S_n(T) \times S_n(T)$ is closed, and $[d(\bar{x},\bar{y}) \le r] \subseteq S_{2n}(T)$ is closed, whereby so is its image $\{(p,q)\colon d(p,q) \le r\} \subseteq S_n(T)^2$. □

d-isolated types

Definition

A type $p \in S_n(T)$ is d-isolated if for all $r > 0$ the metric ball $B(p, r)$ contains p in its topological interior: $p \in B(p, r)^\circ$ (i.e., the metric and the topology coincide at p).

Fact

A type $p \in S_n(T)$ is d-isolated if and only if it is weakly d-isolated, i.e., iff for all $r > 0$: $B(x, r)^\circ \neq \varnothing$.

Omitting and realising types in models

Proposition (Henson)

A d-isolated type p is realised in every model of T. If T is countable, then the converse is also true.

Proof.

$\Longrightarrow$ As $B(p, 2^{-n})^\circ \neq \varnothing$ for all n, it must be realised in M, say by a_n. We can furthermore arrange that $d(a_n, a_{n+1}) < 2^{-n-1}$. Then $a_n \to a \models p$.

$\Longleftarrow$ If $\bar{B}(p, r)^\circ = \varnothing$ for some $r > 0$, we can omit it in a dense subset of M. Then M omits p. □

Ryll-Nardzewski Theorem

Definition

A theory T is λ-categorical if for all $M, N \models T$:

$$\|M\| = \|N\| = \lambda \Longrightarrow M \simeq N.$$

Theorem (Henson)

For a complete countable theory T, TFAE:

- *T is $\aleph_0$-categorical (unique separable model).*
- *Every n-type over $\varnothing$ is d-isolated for all n.*
- *The metric and topology coincide on each $(S_n(T), d)$.*
- *Every automorphism-invariant uniformly continuous predicate on $\bar{M}$ is definable.*

Outline

Stable theories

Recall: $\| \cdot \|$ denotes the metric density character.

Definition

- (Iovino) We say that T is λ-stable if $\|A\| \leq \lambda \Longrightarrow \| S_n(A)\| \leq \lambda$.
- It is stable if it is λ-stable for some λ.
- It is superstable if it is λ-stable for all λ big enough.

Proposition

The following are equivalent:

- *T is stable.*
- *If $\|M\| \leq 2^{|T|}$ then $|S_n(M)| \leq 2^{|T|}$.*

Notions of independence

Let $\bar{M}$ be a monster model, and $\mathop{\smile\!\!\!\!\!\mid}$ a ternary notion of independence between small subsets of $\bar{M}$: $A \mathop{\smile\!\!\!\!\!\mid}_B C$ means "A is independence from C over B." It may satisfy:

- Invariance: under automorphisms of $\bar{M}$.
- Symmetry: $A \mathop{\smile\!\!\!\!\!\mid}_B C \iff C \mathop{\smile\!\!\!\!\!\mid}_B A$.
- Transitivity: $A \mathop{\smile\!\!\!\!\!\mid}_B CD \iff \left[A \mathop{\smile\!\!\!\!\!\mid}_B C \text{ and } A \mathop{\smile\!\!\!\!\!\mid}_{BC} D\right]$.
- Finite character: $A \mathop{\smile\!\!\!\!\!\mid}_B C \iff \bar{a} \mathop{\smile\!\!\!\!\!\mid}_B C$ for all finite $\bar{a} \in A$.

Notions of independence

Let $\bar{M}$ be a monster model, and $\downarrow$ a ternary notion of independence between small subsets of $\bar{M}$: $A \downarrow_B C$ means "A is independence from C over B." It may satisfy:

- Invariance: under automorphisms of $\bar{M}$.
- Symmetry: $A \downarrow_B C \iff C \downarrow_B A$.
- Transitivity: $A \downarrow_B CD \iff \left[A \downarrow_B C \text{ and } A \downarrow_{BC} D\right]$.
- Finite character: $A \downarrow_B C \iff \bar{a} \downarrow_B C$ for all finite $\bar{a} \in A$.
- Extension: for all $\bar{a}$, B, C there is $\bar{a}'$ such that $\operatorname{tp}(\bar{a}/B) = \operatorname{tp}(\bar{a}'/B)$ and $\bar{a}' \downarrow_B C$.
- Local character: For all $\bar{a}$ and B there is $B_0 \subseteq B$ such that $|B_0| \leq |T|$ and $\bar{a} \downarrow_{B_0} B$.
- Stationarity: if $M \preceq \bar{M}$, $\bar{a} \downarrow_M B$, $\bar{a}' \downarrow_M B$ then: $\operatorname{tp}(\bar{a}/M) = \operatorname{tp}(\bar{a}'/M) \implies \operatorname{tp}(\bar{a}/B) = \operatorname{tp}(\bar{a}'/B)$.

Stability and independence

Theorem

T is stable if and only if its monster models admit notions of independence satisfying all of the above. Moreover, if such a notion of independence exists then it is unique.

Example

- In Hilbert spaces: $\perp\!\!\!\perp$ = orthogonality.
- In probability algebras: $\perp\!\!\!\perp$ = probabilistic independence.
- In L^p lattices: more complicated.

- Itaï Ben-Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Expanded lecture notes for a workshop given in March/April 2005, Isaac Newton Institute, University of Cambridge.
- Itaï Ben-Yaacov and Alexander Usvyatsov, *Continuous first order logic and local stability*, submitted.