General Conjectures and Problems

Dichotomy Conjecture: Given a constraint language Γ, either CSP(Γ) is in P or it is NP-complete.

Algebraic Dichotomy Conjecture: Let Γ be a constraint language with an idempotent associated algebra $\mathbb{A}(\Gamma)$. If the variety generated by $\mathbb{A}(\Gamma)$ omits the unary type then CSP(Γ) is tractable.

Definability-in-Datalog Conjecture: Let Γ be a constraint language with an idempotent associated algebra $\mathbb{A}(\Gamma)$. Then CSP(Γ) is tractable via definability in Datalog if and only if the variety generated by $\mathbb{A}(\Gamma)$ omits the unary and affine types.

Definability-in-fragments-of-Datalog Conjectures: Let Γ be a constraint language with an idempotent associated algebra $\mathbb{A}(\Gamma)$. Then

- CSP(Γ) is tractable via definability in linear Datalog if and only if the variety generated by $\mathbb{A}(\Gamma)$ omits the unary, affine and semilattice types;
- CSP(Γ) is tractable via definability in symmetric Datalog if and only if the variety generated by $\mathbb{A}(\Gamma)$ omits the unary, affine, semilattice and lattice types.

Dichotomy Conjecture for the Descriptive Complexity of CSP: For every constraint language Γ, either CSP(Γ) is definable in Datalog or CSP(Γ) is not definable in the infinitary logic $C^{\omega}_{\infty\omega}$, a powerful finite-variable infinitary logic with counting quantifiers.

The Hierarchy Problem for Datalog-definable CSPs: Prove or disprove that for every $k \geq 4$, there is a constraint language Γ_k such that CSP(Γ_k) is definable in k-Datalog, but not in $(k-1)$-Datalog.

Datalog vs. Least Fixed-Point Logic for CSP: Is there a constraint language Γ such that CSP(Γ) is definable in least fixed point logic LFP (hence, it is tractable), but not definable in Datalog?

A Logic for Tractable CSP: Is there a logic L such that the following two properties hold:

1. The data complexity of L is in PTIME.
2. L can express every polynomial-time solvable CSP(Γ).

The Complexity of the ∃-Inv Sat Problem: Identify the computational complexity of the following structure identification problem: given a relation R and a finite set of relations S over the same domain, is R definable by a conjunctive query (i.e., a primitive positive formula) over S?

Note that this problem is known to be solvable in quadratic time over the Boolean domain; its exact complexity, however, is not known over domains with more than two elements.

Tractability/Finite Width Meta-Problems: Determine the computational complexity of the meta-problems associated with the Algebraic Dichotomy Conjecture and the Definability-in-Datalog Conjecture. In other words, given a finite constraint language Γ, how difficult is it to determine if the variety generated by the associated algebra $\mathbb{A}(\Gamma)$ omits the unary (or the unary and affine) type(s). Bulatov has shown that both of these problems are NP-hard, and it is the case that they are in the class Π_2. Are they Π_2-complete or NP-complete?

Local versus Global Tractability: To date, all infinite constraint languages that are known to be tractable are in fact globally tractable. Determine whether or not tractability is equivalent to global tractability.

More specific Problems and Conjectures

- Are there finite constraint languages with at most binary (or at most ternary) relations that can be solved by some bounded width but not by width three (or four)? Find candidates on which to test the methods for proving lower bounds on refutation complexity of random instances, and understand where they fail when they do.
- Investigate algorithmic methods that are potentially stronger than bounded width, such as bounded Sherali-Adams rank in mathematical programming, or logics with counting in descriptive complexity.
- Is there an ω-categorical structure $\mathbb{B}$ such that CSP($\mathbb{B}$) is in NP, but neither NP-complete nor in P?
- Can we characterize all those structures $\mathbb{B}$ where primitive positive definability is captured by preservation under polymorphisms ?
- Classify the constraint satisfaction problems for all structures with a first-order definition in $(\mathbb{C}, +, *)$ (the complex numbers with addition and multiplication).
- Study the existence (and formulation) of an algebraic characterization of path and pathwidth dualities (e.g. an equivalent notion to that of relational width).
- Determine a complete list of those polymorphisms that imply bounded treewidth and bounded pathwidth dualities.
- Bulatov has shown that list constraints (or conservative constraints) are in P or are NP-complete. Feder and Hell have shown that this classification still holds if not all lists are required, only lists of size at most three and optionally other lists. The obvious question is the case where only the lists of size at most two and optionally other lists are required. If all pairs were generalized-majority-minority or all pairs were commutative-conservative this is easy to show from existing results, but when all three kinds of pairs occur the problem remains open.
- Feder showed that a general subgroup problem can be extended to a near subgroup problem and remain Maltsev, while a non-near-subgroup extension is NP-complete by Feder and Vardi. The obvious question is whether when we start with the general case of a Maltsev problem and extend it to a non-Maltsev problem it becomes NP-complete. The same can be asked more generally for extending a gmm to a non-gmm giving NP-completeness.

- construct, or otherwise determine, several small or simple examples of CSP's whose complexity/expressibility is still undetermined.
- Prove/disprove: there exists a constant L such that, if CSP(Γ) admits an nuf (of any arity) then ¬CSP(Γ) is in (L, K)-Datalog for some K. What about $L = 2$?
- An algebra $\mathbb{A}$ is *tractable* if every CSP(Γ) with finite Γ invariant under the operations of $\mathbb{A}$ is tractable. Let $\mathbb{A}$ and $\mathbb{B}$ be tractable algebras of the same signature. Prove that the product $\mathbb{A} \times \mathbb{B}$ is also tractable. Same question, replacing "tractable" by "in L", "in NL", etc.
- Prove or disprove: if Γ is a finite set of relations that are invariant under Jónsson terms, then they are invariant under a near-unanimity operation. This is known to be true for Boolean domains and conjectured (by Zadori) to be true in general.
- Prove that constraint languages that are invariant under a sequence of Jónsson terms are of bounded width. In particular, show this for sequences of Jónsson terms of length 6.
- Generalise the Barto, Kozik, Niven dichotomy result on graphs without sinks and sources to other types of graphs.
- Maroti and McKenzie have shown that if $\mathbb{A}$ is a finite algebra such that the variety generated by $\mathbb{A}$ omits the unary and affine types then for some $N > 0$ and for every $k > N$, $\mathbb{A}$ has a weak near unanimity term of arity k. For each $N > 0$ construct a finite algebra that generates a variety that omits the unary and affine types and that does not have a weak near unanimity term of arity N; or establish that for some N no such algebra exists. This could help to resolve the Hierarchy Problem for Datalog-Definable CSPs.
- Investigate the tractability of constraint languages that do not fall within the scope of the two algorithmic paradigms arising from Datalog-definability or having few subpowers. In particular focus on languages that have Gumm terms as polymorphisms. (An algebra $\mathbb{A}$ has Gumm terms if and only if it generates a variety whose members have congruence lattices that satisfy the modular law.) According to the Algebraic Dichotomy Conjecture, such languages should be tractable.
- Determine the computational complexity of deciding, given a constraint language Γ, if the counting-CSP(Γ) problem is tractable. By a theorem of Bulatov, this amounts to determining if an associated algebra generates a congruence singular variety or not. Congruence singularity is a property of a congruence lattice that is related to congruence permutability and congruence uniformity.
- How can the presence of polymorphisms that exhibit some type of symmetry amongst its variables be used to establish tractability? For instance, Barto, Kozik, Maroti, McKenzie, and Niven have shown that constraint languages that have Jonsson terms or, more generally, Gumm terms as polymorphisms have cyclic polymorphisms, i.e., idempotent polymorphisms that satisfy the cyclic identity $t(a, b, ..., , x, y, z) = t(z, a, b, c, ..., x, y)$. Of what value is having cyclic polymorphisms vis-a-vis tractability? What other constraint languages have cyclic polymorphisms?
- Is the problem of determining if a finite constraint language has a near unanimity operation as a polymorphism decidable? If so, what is its computational complexity?

- Prove the tractability of finite algebras with a weak near-unanimity term satisfying $t(y, x, ..., x) \in \{x, y\}$ for all x, y. This would be a generalization of the conservative algebra result of A. Bulatov, and the generalized majority-minority term result of V. Dalmau.
- Describe the structure of core oriented trees that have a weak near-unanimity polymorphism. Is it true that all such trees have bounded width?
- Bulatov introduced colored graphs of (idempotent) algebras and proved that the connectivity of such graphs and omitting edges of the affine type (+connectivity) is equivalent to omitting type 1, and types 1, 2, respectively. Are there any other connections to the studied properties of algebras?
- Search for complexity classes such that problems complete in those classes are representable as CSP(Γ), CSP($\mathbb{A}$) for some constraint language Γ or and algebra $\mathbb{A}$. Are there any properties except for dualities responsible for the complexity of a problem?
- Classify the complexity of the QCSP (Quantified CSP) and associated universal algebra.
- Is there an approach to use directly the properties of congruences to get tractability results, rather than using the corresponding polymorphisms of a constraint language?
- Identify structural properties of relations using the properties of the congruence lattices of their algebras.