A Universal Algebra Primer for CSP

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The Paradigm

- Every Constraint Satisfaction Problem $=$ a homomorphism problem $CSP(\mathbf{B})$ for some relational structure $\mathbf{B}$;
- to each structure $\mathbf{B}$ is associated an algebra $\mathbb{A}(\mathbf{B})$;
- the equational properties of $\mathbb{A}(\mathbf{B})$ control the descriptive and algorithmic complexity of the decision problem $CSP(\mathbf{B})$.

Constraint Satisfaction Problems

Let $\Gamma = \{\theta_1, \ldots, \theta_r\}$ be a constraint language, i.e. a set of relations on a finite set A.

The decision problem $CSP(\Gamma)$:

- **Instance**: *variables* $x_1, \ldots, x_s$ and *constraints* $[(x_{i_1}, x_{i_2}, \ldots, x_{i_k}), \theta_j], [(x_{l_1}, x_{l_2}, \ldots, x_{l_m}), \theta_v], \ldots$
- **Question**: can one assign values $a_1, \ldots, a_s$ to $x_1, \ldots, x_s$ such that all constraints are satisfied, i.e. $(a_{i_1}, a_{i_2}, \ldots, a_{i_k}) \in \theta_j, (a_{l_1}, a_{l_2}, \ldots, a_{l_m}) \in \theta_v, \ldots$?

Our goal:
to determine the algorithmic/descriptive complexity of $CSP(\Gamma)$.

Homomorphism Problems

Convenient to view $CSP(\Gamma)$ as follows:

Let $\mathbf{B} = \langle A; \theta_1, \dots, \theta_r \rangle$ be the relational structure on A whose basic relations are those in Γ;

$$CSP(\mathbf{B}) = \{\mathbf{C} : \mathbf{C} \to \mathbf{B}\}$$

$\mathbf{C} \to \mathbf{B}$ means "there exists a homomorphism from **C** to **B**"
i.e.
if $\mathbf{C} = \langle X; \rho_1, \dots, \rho_r \rangle$, there is a map $f : X \to A$ such that $f(\rho_i) \subseteq \theta_i$ for all i.

Both formulations are equivalent (Feder, Vardi)

Core structures

If **B** and $\mathbf{B}_0$ are homomorphically equivalent, i.e.
if $\mathbf{B}_0 \to \mathbf{B}$ and $\mathbf{B} \to \mathbf{B}_0$, then

$$CSP(\mathbf{B}_0) = CSP(\mathbf{B}).$$

Hence we may always assume **B** is a core,
i.e.
B has no proper retracts,
i.e.
every homomorphism from **B** to **B** is onto,
i.e.
of all structures equivalent to **B**, **B** has smallest universe.

The Dichotomy Conjecture

Dichotomy Conjecture (Feder-Vardi, 1993)

Every $CSP(\mathbf{B})$ is either in P or NP-complete.

Part 1: the Algebra associated to a CSP

Relational structure **B**

$\downarrow$

set of relations

$\updownarrow$

set of operations

$\downarrow$

algebra $\mathbb{A}(\mathbf{B})$

A Fundamental Duality

Relational structure **B**

$\downarrow$

set of relations

Fundamental duality $\leadsto$ $\updownarrow$

set of operations

$\downarrow$

algebra $\mathbb{A}(\mathbf{B})$

A Fundamental Duality, cont'd

Let A be a finite set.

- Let $f : A^n \longrightarrow A$ be an n-ary operation on A;
- Let $\theta \subseteq A^k$ be a k-ary relation on A.

A Fundamental Duality, cont'd

Let A be a finite set.

- Let $f : A^n \to A$ be an n-ary operation on A;
- Let $\theta \subseteq A^k$ be a k-ary relation on A.
- The operation f preserves the relation θ, or θ is invariant under f, if the following holds:

$$\underbrace{\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ldots & \vdots \\ a_{k,1} & \cdots & a_{k,n} \end{bmatrix}}_{\text{columns in } \theta} \stackrel{f}{\mapsto} \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix}}_{\theta}$$

Applying f to the rows of the matrix with columns in θ yields a tuple of θ.

A Fundamental Duality, cont'd

Example

On $\{0,1\}$ let $\leq$ denote the usual ordering $\{(0,0),(0,1),(1,1)\}$.
An operation f preserves $\leq$ iff it is *monotonic*, i.e.
$f(x_1,\ldots,x_n) \leq f(y_1,\ldots,y_n)$ whenever $x_i \leq y_i$ for all $1 \leq i \leq n$.

$$\begin{bmatrix} x_1 & \cdots & x_n \\ \text{|}\!\wedge & \cdots & \text{|}\!\wedge \\ y_1 & \cdots & y_n \end{bmatrix} \overset{f}{\longmapsto} \begin{bmatrix} f(x_1,\ldots,x_n) \\ \text{|}\!\wedge \\ f(y_1,\ldots,y_n) \end{bmatrix}$$

Clones and Relational Clones

- Let Γ be a set of relations on A.

$$Pol(\Gamma) = \{f : f \text{ preserves every } \theta \in \Gamma\};$$

sets of operations of this form are clones.

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- Let F be a set of operations on A.

$$Inv(F) = \{\theta : \theta \text{ is invariant under every } f \in F\};$$

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  sets of relations of this form are relational clones.

- $Pol(Inv(F))$ is the clone generated by F;
  $Inv(Pol(\Gamma))$ is the relational clone generated by Γ.

Clones ...

A constructive view of clones:

- g is in the clone generated by F iff it is obtained from members of F and projections by composition:

$$g(x,y,z,t) = f_1(x, f_2(y,x), f_3(f_2(t,z),y))$$

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Example

On $\{0,1\}$, the clone of monotonic functions $Pol(\leq)$ is generated by $\{\vee,\wedge,0,1\}$: any non-constant monotonic function is of the form

$$f(x_1,\dots,x_n) = (x_{i_1} \vee x_{i_2} \cdots \vee x_{i_s}) \wedge (\cdots) \wedge \cdots$$

... and Relational Clones

A constructive view of relational clones:

- ρ is in the relational clone generated by Γ iff it is (primitive positive) PP-definable from relations in Γ:

$$\rho = \{(x, y, z) : \exists u \, \exists v \, \Phi(x, y, z, u, v)\}$$

Φ is a conjunct of atomic formulas from relations in Γ,

$$\Phi(x, y, z, u, v) \equiv (x, u) \in \gamma_1 \wedge (y, v, z, z) \in \gamma_2.$$

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Example

On $\{0, 1\}$, the relational clone generated by $\leq$ consists of all relations of the form

$$\theta = \{(x_1, \ldots, x_k) : x_{i_1} \leq x_{j_1}, \ldots, x_{i_s} \leq x_{j_s}\}.$$

The Definition of the Algebra $\mathbb{A}(\mathbf{B})$

Relational structure $\mathbf{B}$

$\downarrow$

set of relations

$\updownarrow$

set of operations

definition of $\rightsquigarrow$ $\downarrow$

algebra $\mathbb{A}(\mathbf{B})$

$\mathbb{A}$lgebras

Let A be a non-void set.

- A (non-indexed) algebra is a pair $\mathbb{A} = \langle A; F \rangle$ where F is a set of operations on A, the basic or fundamental operations of $\mathbb{A}$.

$\mathbb{A}$lgebras

Let A be a non-void set.

- A (non-indexed) algebra is a pair $\mathbb{A} = \langle A; F \rangle$ where F is a set of operations on A, the basic or fundamental operations of $\mathbb{A}$.
- The members of the clone generated by F are called the term operations of $\mathbb{A}$.
- If we also allow the use of constant operations, we obtain the polynomial operations of $\mathbb{A}$.
- Algebras on the same universe with the same term (polynomial) operations are term (polynomially) equivalent.

Term equivalent Algebras

Example

Let $\mathcal{O}$ denote the set of all operations on $\{0,1\}$.
$\mathbb{A} = \langle\{0,1\}; \wedge, \vee, \neg\rangle$ and $\mathbb{B} = \langle\{0,1\}; \mathcal{O}\rangle$ are term equivalent.

Term equivalent Algebras

Example

Let $\mathcal{O}$ denote the set of all operations on $\{0,1\}$.
$\mathbb{A} = \langle\{0,1\};\wedge,\vee,\neg\rangle$ and $\mathbb{B} = \langle\{0,1\};\mathcal{O}\rangle$ are term equivalent.

Example

- Let $m(x,y,z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$, the *majority* operation on $\{0,1\}$.
- $\mathbb{A} = \langle\{0,1\};\wedge,\vee\rangle$ and $\mathbb{B} = \langle\{0,1\};m\rangle$ are *not* term equivalent, but they are polynomially equivalent.

The Algebra $\mathbb{A}(\mathbf{B})$

Let $\mathbf{B} = \langle A; \Gamma \rangle$ be a relational structure.

The algebra associated to **B** is

$$\mathbb{A}(\mathbf{B}) = \langle A; Pol(\Gamma) \rangle.$$

The Algebra $\mathbb{A}(\mathbf{B})$, cont'd

Example

- Let $\mathbf{B} = \langle \{0,1\}; \leq, \{0\}, \{1\} \rangle$.
- $\mathbb{A}(\mathbf{B}) = \langle \{0,1\}; Pol(\leq, \{0\}, \{1\}) \rangle$.
- The term (basic) operations of $\mathbb{A}(\mathbf{B})$ are all monotonic Boolean operations f such that $f(0,\ldots,0) = 0$ and $f(1,\ldots,1) = 1$.

Idempotent Algebras

- It is convenient to consider idempotent algebras, i.e. algebras whose basic operations satisfy

$$f(x, \ldots, x) = x \text{ for all } x;$$

- f is idempotent iff it preserves every one-element unary relation $\{a\}$;
- The full idempotent reduct of the algebra $\mathbb{A}(\mathbf{B}) = \langle A; Pol(\Gamma)\rangle$ is the algebra $\langle A; Pol(\Gamma \cup \{\{a\} : a \in A\})\rangle$;
- The term operations of the full idempotent reduct are precisely the *idempotent* term operations of $\mathbb{A}(\mathbf{B})$.

Part 2: The Equational Properties of $\mathbb{A}(\mathbf{B})$

Relational structure **B**

$\downarrow$

set of relations

$\updownarrow$

set of operations

$\downarrow$

algebra $\mathbb{A}(\mathbf{B})$

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algebra $\mathbb{A}(\mathbf{B})$

$\downarrow$

variety $\mathcal{V}(\mathbb{A}(\mathbf{B}))$

Similar Algebras; Identities

- Let I be a set of *operation symbols*, each with a given *arity*;
- $\mathbb{A} = \langle A; q^{\mathbb{A}}(q \in I)\rangle$ is an indexed algebra;
- $q^{\mathbb{A}}$ is the *interpretation* of q in $\mathbb{A}$, and arities match.
- An algebra $\mathbb{C} = \langle C; q^{\mathbb{C}}(q \in I)\rangle$ is *similar* to $\mathbb{A}$.

Similar Algebras; Identities, cont'd

- An identity is an expression of the form

$$s(x_1,\ldots,x_n)\approx t(x_1,\ldots,x_n)$$

where s, t are *terms*, e.g.

$$F(x,G(y,z),t)\approx F_1(x,F_2(y,x),F_3(F_2(t,z),y))$$

where F,G,F_1,F_2,F_3 are operation symbols.

- Identities interpret in $\mathbb{A}$: if the resulting term operations are equal, the identity holds in $\mathbb{A}$, or $\mathbb{A}$ *models* the identity.

Equational classes

A class of similar algebras is equational if it consists of all algebras that model some set of identities.

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Example

The class of all algebras $\mathbb{S} = \langle S; \wedge^{\mathbb{S}} \rangle$ satisfying the identities

$$\begin{aligned} x \wedge (y \wedge z) &\approx (x \wedge y) \wedge z, \\ x \wedge y &\approx y \wedge x, \\ x \wedge x &\approx x, \end{aligned}$$

is the (equational) class of semilattices.

Varieties

Theorem (G. Birkhoff)

A class of similar algebras is an equational class iff it is a variety, i.e. if it is closed under the formation of products, subalgebras and homomorphic images.

H, S, and P

- (P) $\mathbb{A} \times \mathbb{C} = \langle A \times C; q^{\mathbb{A}\times\mathbb{C}}\ (q \in I)\rangle$ where $q^{\mathbb{A}\times\mathbb{C}}$ acts coordinatewise as $q^{\mathbb{A}}$ and $q^{\mathbb{C}}$.

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- (P) $\mathbb{A} \times \mathbb{C} = \langle A \times C; q^{\mathbb{A}\times\mathbb{C}} \, (q \in I) \rangle$ where $q^{\mathbb{A}\times\mathbb{C}}$ acts coordinatewise as $q^{\mathbb{A}}$ and $q^{\mathbb{C}}$.
- (S) $\emptyset \neq C \subseteq A$ is a *subuniverse* if it invariant under the basic operations of $\mathbb{A}$:
  $\mathbb{C} = \langle C; q^{\mathbb{C}} \, (q \in I) \rangle$ where $q^{\mathbb{C}}$ is the restriction of $q^{\mathbb{A}}$ to C;

H, S, and P

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- (S) $\emptyset \neq C \subseteq A$ is a *subuniverse* if it invariant under the basic operations of $\mathbb{A}$:
  $\mathbb{C} = \langle C; q^{\mathbb{C}} (q \in I)\rangle$ where $q^{\mathbb{C}}$ is the restriction of $q^{\mathbb{A}}$ to C;
- (H) if α is a congruence of $\mathbb{A}$, i.e. a partition of A invariant under the operations of $\mathbb{A}$:
  $\mathbb{A}/\alpha = \langle A/\alpha\,;\, q^{\mathbb{A}/\alpha} (q \in I)\rangle$ where $q^{\mathbb{A}/\alpha}$ is the action of $q^{\mathbb{A}}$ on the α-blocks.
  (Every image of the algebra $\mathbb{A}$ under a homomorphism is of this form: α is the *kernel* of the homomorphism.)

HSP

Theorem (Tarski)

Let $\mathcal{V}(\mathcal{A})$ denote the variety generated by the class of algebras $\mathcal{A}$. Then $\mathcal{V}(\mathcal{A}) = HSP(\mathcal{A})$.

HSP

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Example

Let $\mathbb{S} = \langle \{0, 1\}; \wedge \rangle$ denote the 2-element (meet) semilattice. $HSP(\mathbb{S})$ is the variety of semilattices, i.e. every semilattice is a homomorphic image of a subalgebra of a power of $\mathbb{S}$.

Part 3: Linking the Complexity of $CSP(\mathbf{B})$ to the Equational Properties of $\mathbb{A}(\mathbf{B})$

Core relational structure **B**

$\downarrow$

algebra $\mathbb{A}(\mathbf{B})$

$\downarrow$

full idempotent reduct $\mathbb{A}$ of $\mathbb{A}(\mathbf{B})$

$\downarrow$

variety $\mathcal{V}(\mathbb{A})$

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- Let $\mathbb{C} = \langle C; G \rangle \in \mathcal{V}(\mathbb{A})$ be finite.

Part 3: Linking the Complexity of $CSP(\mathbf{B})$ to the Equational Properties of $\mathbb{A}(\mathbf{B})$

$$\begin{array}{c}\text{Core relational structure } \mathbf{B} \\ \downarrow \\ \text{algebra } \mathbb{A}(\mathbf{B}) \\ \downarrow \\ \text{full idempotent reduct } \mathbb{A} \text{ of } \mathbb{A}(\mathbf{B}) \\ \downarrow \\ \text{variety } \mathcal{V}(\mathbb{A})\end{array}$$

- Let $\mathbb{C} = \langle C; G \rangle \in \mathcal{V}(\mathbb{A})$ be finite.
- Let $\mathbf{B}_0 = \langle C; \Gamma_0 \rangle$ such that $\Gamma_0 \subseteq Inv(G)$.

Part 3: Linking the Complexity of $CSP(\mathbf{B})$ to the Equational Properties of $\mathbb{A}(\mathbf{B})$

Core relational structure $\mathbf{B}$

$\downarrow$

algebra $\mathbb{A}(\mathbf{B})$

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full idempotent reduct $\mathbb{A}$ of $\mathbb{A}(\mathbf{B})$

$\downarrow$

variety $\mathcal{V}(\mathbb{A})$

- Let $\mathbb{C} = \langle C; G\rangle \in \mathcal{V}(\mathbb{A})$ be finite.
- Let $\mathbf{B}_0 = \langle C; \Gamma_0\rangle$ such that $\Gamma_0 \subseteq Inv(G)$.
- How are $CSP(\mathbf{B}_0)$ and $CSP(\mathbf{B})$ related ?

Algebraic Reductions: Algorithmic Complexity

Theorem (Jea + Bul-Jea-Kro + Ats-Bul-Daw + Lar-Tes)

Let $\mathbf{B}$ *be a core relational structure. Let* $\mathbb{A}$ *denote the full idempotent reduct of* $\mathbb{A}(\mathbf{B})$.

- *Let* $\mathbb{C}$ *be a finite algebra in* $\mathcal{V}(\mathbb{A})$*, and let* $\mathbf{B}_0$ *be a relational structure whose relations are invariant under the basic operations of* $\mathbb{C}$*. Then there exists a logspace many-one reduction of* $CSP(\mathbf{B}_0)$ *to* $CSP(\mathbf{B})$.
- *If furthermore* $\mathbb{C} \in HS(\mathbb{A})$ *and the relations of* $\mathbf{B}_0$ *are irredundant, then the above reduction is first-order and without ordering.*

Algebraic Reductions: Descriptive Complexity

Theorem (Ats-Bul-Daw + Lar-Zád + Lar-Tes)

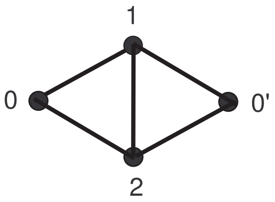
Let $\mathbf{B}$ *be a core relational structure. Let* $\mathbb{A}$ *denote the full idempotent reduct of* $\mathbb{A}(\mathbf{B})$.

- *Let* $\mathbb{C}$ *be a finite algebra in* $\mathcal{V}(\mathbb{A})$*, and let* $\mathbf{B}_0$ *be a relational structure whose relations are invariant under the basic operations of* $\mathbb{C}$.
  If $\neg CSP(\mathbf{B})$ *is expressible in (linear, symmetric) Datalog, then so is* $\neg CSP(\mathbf{B}_0)$.

(Datalog: see Phokion Kolaitis' talk)

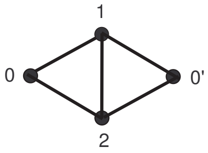
Algebraic Reductions: an Illustration

A graph on $A = \{0, 0', 1, 2\}$ with (irreflexive, symmetric) edge relation θ:



Let $\mathbf{B} = \langle A; \theta, \{0\}, \{0'\}, \{1\}, \{2\}\rangle$.

Example 1, cont'd

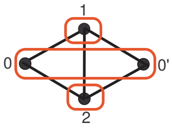


- $\mathbb{A} = \mathbb{A}(\mathbf{B})$ is an idempotent algebra.
- It has the following proper subuniverses:

$$\{0, 0'\} = \{x : (x, 1) \in \theta \wedge (x, 2) \in \theta\},$$

$$\{1, 2\} = \{x : (x, 0) \in \theta \wedge (x, 0') \in \theta\}.$$

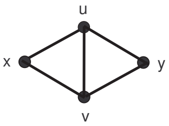
Example 1, cont'd



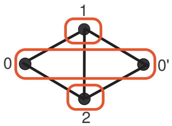
The following is a congruence of $\mathbb{A}$:

$$\alpha = \{(x, y) : \exists u,\, v,\, \Phi(x, y, u, v)\},$$

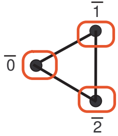
where Φ is given by:



Example 1, cont'd

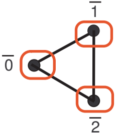


The operations of $\mathbb{A}/\alpha$ preserve $\overline{\theta}$ induced by θ:



If $\mathbf{B}_0 = \langle\{\overline{0}, \overline{1}, \overline{2}\}; \overline{\theta}\rangle$, then $CSP(\mathbf{B}_0)$ reduces to $CSP(\mathbf{B})$.

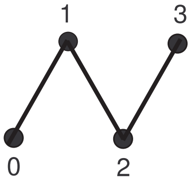
Example 1, cont'd



Since $CSP(\mathbf{B}_0)$ is 3-colouring, we conclude that $CSP(\mathbf{B})$ is NP-complete.

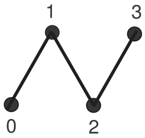
Algebraic Reductions: another Illustration

A *reflexive* graph on $A = \{0, 1, 2, 3\}$ with symmetric edge relation θ:



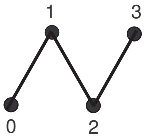
Let $\mathbf{B} = \langle A; \theta, \text{ all subsets of } A\rangle$.
(A *list-homomorphism problem*.)

Example 2, cont'd



- $\mathbb{A} = \mathbb{A}(\mathbf{B})$ is a conservative algebra, i.e. every subset is a subuniverse.
- The algebra $\mathbb{A}$ has a majority term (Feder, Hell, Huang) and hence $CSP(\mathbf{B})$ is in NL (Dalmau, Krokhin).
- Can we do better ?

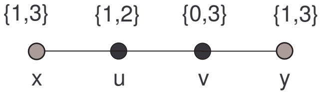
Example 2, cont'd



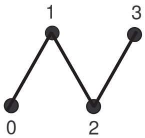
The operations of the subalgebra $\mathbb{C}$ with universe $\{1,3\}$ preserve

$$\rho = \{(x,y) : \exists u,\, v,\, \Phi(x,y,u,v)\},$$

where Φ is given by:



Example 2, cont'd



- We have $\rho = \{(1,1),(1,3),(3,3)\}$, i.e. the natural order on $\{1,3\}$;
- if $\mathbf{B}_0 = \langle\{1,3\};\rho,\{1\},\{3\}\rangle$ then $CSP(\mathbf{B}_0)$ reduces to $CSP(\mathbf{B})$;
- Since $CSP(\mathbf{B}_0)$ is (essentially) directed unreachability, we conclude that $CSP(\mathbf{B})$ is NL-complete.

The Dichotomy Conjecture(s)

Core relational structure **B**

$\downarrow$

full idempotent reduct $\mathbb{A}$ of $\mathbb{A}(\mathbf{B})$

$\downarrow$

variety $\mathcal{V}(\mathbb{A})$

The Dichotomy Conjecture(s)

Core relational structure **B**

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$\downarrow$

variety $\mathcal{V}(\mathbb{A})$

- If $\mathbb{C} \in \mathcal{V}(\mathbb{A})$ is "NP-hard", then $CSP(\mathbf{B})$ is NP-hard.
- All known hardness results for CSP's are of this form ...
- ... so what happens if $\mathcal{V}(\mathbb{A})$ contains *no* "bad algebra" ?

The Dichotomy Conjecture(s), cont'd

Dichotomy Conjecture (Feder-Vardi, 1993)

Every $CSP(\mathbf{B})$ is either in P or NP-complete.

The Dichotomy Conjecture(s), cont'd

Dichotomy Conjecture (Feder-Vardi, 1993)

Every $CSP(\mathbf{B})$ is either in P or NP-complete.

An algebra $\mathbb{C}$ is a *set* if its basic operations are projections.

Tractability Conjecture (Bulatov-Jeavons-Krokhin, 2000)

Let $\mathbf{B}$ be a core structure and let $\mathbb{A}$ be the full idempotent reduct of $\mathbb{A}(\mathbf{B})$. If $\mathcal{V}(\mathbb{A})$ contains no set, then $CSP(\mathbf{B})$ is in P, otherwise it is NP-complete.

Miscellaneous Remarks

- In the examples: “bad algebra” in $\mathcal{V}(\mathbb{A})$ actually in $HS(\mathbb{A})$: true in general (Bul-Jea for sets, Valeriote for general case)
- NP-hardness, NL-hardness and P-hardness are detected by certain 2-element algebras in $HS(\mathbb{A})$;
- preventing “bad algebras” in the variety means “nice equations” ...
- hence: do “nice equations” imply tractability (etc.) ?
- Some evidence: nuf, $CD(4)$, 2-semilattices, TSI, few subpowers, etc.
- See Matt Valeriote’s talk.