Conformal structure in Geometry, Analysis, and Physics

The American Institute of Mathematics

This is a hard–copy version of a web page available through http://www.aimath.org
Input on this material is welcomed and can be sent to `workshops@aimath.org`
Version: Wed Oct 15 14:10:49 2003

Table of Contents

Chapter A: A Primer on Q-Curvature

by Michael Eastwood and Jan Slovák

Disclaimer: These are rough notes only, aimed at setting the scene and promoting discussion at the American Institute of Mathematics Research Conference Center Workshop 'Conformal Structure in Geometry, Analysis, and Physics,' 12th–16th August 2003. For simplicity, we have omitted all references. Curvature conventions are in an appendix. Conversations with Tom Branson and Rod Gover have been extremely useful.

Let M be an oriented even-dimensional Riemannian n-manifold. Branson's Q-curvature is a canonically defined n-form on M. It is not conformally invariant but enjoys certain natural properties with respect to conformal transformations.

When $n = 2$, the Q-curvature is a multiple of the scalar curvature. With conventions as in the appendix $Q = -\frac{1}{2}R$. Under conformal rescaling of the metric, $g_{ab} \mapsto \widehat{g}_{ab} = \Omega^2 g_{ab}$ we have

$$\widehat{Q} = Q + \Delta \log \Omega,$$

where $\Delta = \nabla^a \nabla_a$ is the Laplacian.

When $n = 4$, the Q-curvature is given by

$$Q = \tfrac{1}{6}R^2 - \tfrac{1}{2}R^{ab}R_{ab} - \tfrac{1}{6}\Delta R. \tag{1}$$

Under conformal rescaling,

$$\widehat{Q} = Q + P \log \Omega,$$

where P is the Paneitz operator

$$Pf = \nabla_a \left[\nabla^a \nabla^b + 2R^{ab} - \tfrac{2}{3}Rg^{ab}\right] \nabla_b f. \tag{2}$$

For general even n, the Q-curvature transforms as follows:–

$$\widehat{Q} = Q + P \log \Omega, \tag{3}$$

where P is a linear differential operator from functions to n-forms whose symbol is $\Delta^{n/2}$. It follows from this transformation law that P is conformally invariant. To see this, suppose that

$$\widehat{g}_{ab} = \Omega^2 g_{ab} \quad \text{and} \quad \widehat{\widehat{g}}_{ab} = e^{2f}\widehat{g}_{ab} = (e^f\Omega)^2 g_{ab}.$$

Then

$$\widehat{\widehat{Q}} = \widehat{Q} + \widehat{P} \log e^f = Q + P \log \Omega + \widehat{P} f$$

but also

$$\widehat{\widehat{Q}} = Q + P \log(e^f \Omega) = Q + Pf + P \log \Omega.$$

Therefore, $\widehat{P}f = Pf$. With suitable normalisation, P is the celebrated Graham-Jenne-Mason-Sparling operator. Thus, Q may be regarded as more primitive than P and, therefore, is at least as mysterious.

Even when M is conformally flat, the existence of Q is quite subtle. We can reason as follows. When M is actually flat then Q must vanish. Therefore, in the conformally flat case, locally if we write $g_{ab} = \Omega^2 \eta_{ab}$ where η_{ab} is flat, then (3) implies that

$$Q = \Delta^{n/2} \log \Omega, \tag{4}$$

where Δ is the ordinary Laplacian in Euclidean space with η_{ab} as metric. An immediate problem is to verify that this purported construction of Q is well-defined. The problem is that there is some freedom in writing g_{ab} as proportional to a flat metric. If also $g_{ab} = \widehat{\Omega}^2 \widehat{\eta}_{ab}$, then we must show that

$$\Delta^{n/2} \log \Omega = \widehat{\Delta}^{n/2} \log \widehat{\Omega}.$$

This easily reduces to two facts:–

fact 1:: $\Delta^{n/2}$ is conformally invariant on flat space.
fact 2:: if g_{ab} is itself flat, then $\Delta^{n/2} \log \Omega = 0$.

The second of these is clearly necessary in order that (37) be well-defined. For $n = 2$ it is immediate from (17). For $n \geq 4$ it may be verified by direct calculation as follows. If g_{ab} and η_{ab} are both flat then

$$\nabla_a \Upsilon_b = \Upsilon_a \Upsilon_b - \tfrac{1}{2} \Upsilon^c \Upsilon_c g_{ab}, \tag{5}$$

where $\Upsilon_a = \nabla_a \log \Omega$. Therefore,

$$\nabla_c (\Upsilon^a \Upsilon_a)^k = 2k (\Upsilon^a \Upsilon_a)^{k-1} \Upsilon^a \nabla_c \Upsilon_a = k (\Upsilon^a \Upsilon_a)^k \Upsilon_c$$

and

$$\nabla_b \nabla_c (\Upsilon^a \Upsilon_a)^k = k^2 (\Upsilon^a \Upsilon_a)^k \Upsilon_b \Upsilon_c + k (\Upsilon^a \Upsilon_a)^k (\Upsilon_b \Upsilon_c - \tfrac{1}{2} \Upsilon^a \Upsilon_a g_{bc})$$

whence

$$\Delta (\Upsilon^a \Upsilon_a)^k = k(k + 1 - \tfrac{n}{2}) (\Upsilon^a \Upsilon_a)^{k+1}. \tag{6}$$

Taking the trace of (5) gives

$$\Delta \log \Omega = \nabla^a \Upsilon_a = (1 - \tfrac{n}{2}) \Upsilon^a \Upsilon_a$$

and now (6) gives, by induction,

$$\Delta^{k+1} \log \Omega = k! (1 - \tfrac{n}{2})(2 - \tfrac{n}{2}) \cdots (k + 1 - \tfrac{n}{2}) (\Upsilon^a \Upsilon_a)^{k+1}.$$

In particular, $\Delta^{n/2} \log \Omega = 0$, as advertised.

That $\Delta^{n/2}$ is conformally invariant on flat space is well-known. It may also be verified directly by a rather similar calculation. For example, here is the calculation when $n = 4$. For general conformally related metrics $\widehat{g}_{ab} = \Omega^2 g_{ab}$ in dimension 4,

$$\begin{aligned} \widehat{\Delta}^2 f \;&=\; \Delta^2 f + 2\Upsilon^a \Delta \nabla_a f - 2\Upsilon^a \nabla_a \Delta f \\ &\quad + 4(\nabla^a \Upsilon^b) \nabla_a \nabla_b f - 2(\nabla^a \Upsilon_a) \Delta f - 4\Upsilon^a \Upsilon^b \nabla_a \nabla_b f \\ &\quad + 2(\Delta \Upsilon^a) \nabla_a f - 4(\nabla^a \Upsilon^b) \Upsilon_a \nabla_b f - 4(\nabla^a \Upsilon_a) \Upsilon^b \nabla_b f. \end{aligned}$$

If g_{ab} is flat then the third order terms cancel leaving

$$\begin{aligned} \widehat{\Delta}^2 f \;&=\; \Delta^2 f + 4(\nabla^a \Upsilon^b) \nabla_a \nabla_b f - 2(\nabla^a \Upsilon_a) \Delta f - 4\Upsilon^a \Upsilon^b \nabla_a \nabla_b f \\ &\quad + 2(\Delta \Upsilon^a) \nabla_a f - 4(\nabla^a \Upsilon^b) \Upsilon_a \nabla_b f - 4(\nabla^a \Upsilon_a) \Upsilon^b \nabla_b f. \end{aligned}$$

If $\widehat{g}_{ab}$ is also flat, then (5) implies

$$\nabla^a \Upsilon^b = \Upsilon^a \Upsilon^b - \tfrac{1}{2} \Upsilon^c \Upsilon_c g^{ab} \quad \text{and} \quad \nabla^a \Upsilon_a = -\Upsilon^a \Upsilon_a$$

whence the second order terms cancel and the first order ones simplify:–

$$\widehat{\Delta}^2 f = \Delta^2 f + 2(\Delta \Upsilon^b) \nabla_b f + 2\Upsilon^a \Upsilon_a \Upsilon^b \nabla_b f.$$

But using (5) again,

$$\begin{aligned} \Delta \Upsilon^b \;&=\; \nabla_a (\Upsilon^a \Upsilon^b - \tfrac{1}{2} \Upsilon^c \Upsilon_c g^{ab}) \\ &=\; (\nabla^a \Upsilon_a) \Upsilon^b + (\nabla^a \Upsilon^b) \Upsilon_a - (\nabla^b \Upsilon^a) \Upsilon_a = -\Upsilon^a \Upsilon_a \Upsilon^b \end{aligned}$$

and the first order terms also cancel leaving $\widehat{\Delta}^2 f = \Delta^2 f$, as advertised.

Conundrum: Deduce **fact 2** from **fact 1** or vice versa. Both are consequences of (5). Alternatively, construct a Lie algebraic proof of **fact 2**. There is a Lie algebraic proof of **fact 1**. It corresponds to the existence of a homomorphism between certain generalised Verma modules for $\mathfrak{so}(n+1,1)$.

What about a formula for Q, even in the conformally flat case? We have a recipe for Q, namely (37), but it is not a formula. We may proceed as follows.

If $\widehat{g}_{ab} = \Omega g_{ab}$ and g_{ab} is flat, then (16) implies that

$$\nabla_a \Upsilon_b = -\widehat{\mathrm{P}}_{ab} + \Upsilon_a \Upsilon_b - \tfrac{1}{2} g_{ab} \Upsilon^c \Upsilon_c. \tag{7}$$

Taking the trace yields

$$\Delta \log \Omega = \nabla^a \Upsilon_a = -\widehat{\mathrm{P}} - \tfrac{1}{2}(n-2)\Upsilon^a \Upsilon_a. \tag{8}$$

This identity is also valid when $n = 2$: it is (17). Dropping the hat gives $Q = -\mathrm{P} = \frac{1}{2}R$. This is the simplest of the desired formulae.

To proceed further we need two identities. If ϕ has conformal weight w, then as described in the appendix,

$$\widehat{\nabla}_a \phi = \nabla_a \phi + w \Upsilon_a \phi,$$

which we rewrite as

$$\nabla_a \phi = \widehat{\nabla}_a \phi - w \Upsilon_a \phi. \tag{9}$$

Similarly, if ϕ_a has weight w, then

$$\nabla^a \phi_a = \widehat{\nabla}^a \phi_a - (n + w - 2)\Upsilon^a \phi_a \tag{10}$$

and, if ϕ_{ab} is symmetric and has weight w, then

$$\nabla^a \phi_{ab} = \widehat{\nabla}^a \phi_{ab} - (n + w - 2)\Upsilon^a \phi_{ab} + \Upsilon_b \phi^a{}_a. \tag{11}$$

The quantities in (8) have weight -2. Therefore, applying (9) gives

$$\nabla_a \Delta \log \Omega = -\widehat{\nabla}_a \widehat{\mathrm{P}} - 2\Upsilon_a \widehat{\mathrm{P}} - (n-2)\Upsilon^b \nabla_a \Upsilon_b$$

wherein we may use (7) to replace $\nabla_a \Upsilon_b$ to obtain

$$\nabla_a \Delta \log \Omega = -\widehat{\nabla}_a \widehat{\mathrm{P}} - 2\Upsilon_a \widehat{\mathrm{P}} + (n-2)\Upsilon^b \widehat{\mathrm{P}}_{ab} - \tfrac{1}{2}(n-2)\Upsilon_a \Upsilon^b \Upsilon_b.$$

We may now apply ∇^a, using (9), (10), and (11) to replace ∇^a by $\widehat{\nabla}^a$ on the right hand side and (7) to replace derivatives of Υ_a. We obtain an expression involving only complete contractions of $\widehat{\mathrm{P}}_{ab}$, its hatted derivatives, and Υ_a:–

$$\begin{aligned} \Delta^2 \log \Omega \quad = \quad & -\widehat{\Delta}\widehat{\mathrm{P}} - (n-2)\widehat{\mathrm{P}}^{ab}\widehat{\mathrm{P}}_{ab} + 2\widehat{\mathrm{P}}^2 \\ & + (n-6)\Upsilon^a \widehat{\nabla}_a \widehat{\mathrm{P}} + (n-2)\Upsilon^a \widehat{\nabla}^b \widehat{\mathrm{P}}_{ab} + 2(n-4)\Upsilon^a \Upsilon_a \widehat{\mathrm{P}} \\ & - (n-2)(n-4)\Upsilon^a \Upsilon^b \widehat{\mathrm{P}}_{ab} + \tfrac{1}{4}(n-2)(n-4)\Upsilon^a \Upsilon_a \Upsilon^b \Upsilon_b. \end{aligned}$$

Using the Bianchi identity $\widehat{\nabla}^b \widehat{\mathrm{P}}_{ab} = \widehat{\nabla}_a \widehat{\mathrm{P}}$, we may rewrite this as

$$\begin{aligned} \Delta^2 \log \Omega \quad = \quad & -\widehat{\Delta}\widehat{\mathrm{P}} - (n-2)\widehat{\mathrm{P}}^{ab}\widehat{\mathrm{P}}_{ab} + 2\widehat{\mathrm{P}}^2 \\ & + 2(n-4)\Upsilon^a \widehat{\nabla}_a \widehat{\mathrm{P}} + 2(n-4)\Upsilon^a \Upsilon_a \widehat{\mathrm{P}} \\ & - (n-2)(n-4)\Upsilon^a \Upsilon^b \widehat{\mathrm{P}}_{ab} + \tfrac{1}{4}(n-2)(n-4)\Upsilon^a \Upsilon_a \Upsilon^b \Upsilon_b \end{aligned} \tag{12}$$

and, in particular, conclude that when $n = 4$,

$$Q = 2\mathrm{P}^2 - 2\mathrm{P}^{ab}\mathrm{P}_{ab} - \Delta\mathrm{P}. \tag{13}$$

Though it is only guaranteed that this formula is valid in the conformally flat case, in fact it agrees with the general expression (1) in dimension 4.

Of course, we may continue in the vein, further differentiating (38) to obtain a formula for $\Delta^k \log\Omega$ expressed in terms of complete contractions of $\widehat{\mathrm{P}}_{ab}$, its hatted derivatives, and Υ_a. With increasing k, this gets rapidly out of hand. Moreover, it is only guaranteed to give Q in the conformally flat case. Indeed, when $n = 6$ this naive derivation of Q fails for a general metric. Nevertheless, there are already some questions in the conformally flat case.

Conundrum: Find a formula for Q in the conformally flat case. Show that the procedure outlined above produces a formula for Q.

In fact, there is a tractor formula for the conformally flat Q. This is not the place to explain the tractor calculus but, for those who know it already:–

$$\square \begin{bmatrix} n-2 \\ 0 \\ -\mathrm{P} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ Q \end{bmatrix}$$

where

$$\square = D_A \cdots D_B \Big(\Delta - \frac{n-2}{4(n-1)}R\Big) \underbrace{D^B \cdots D^A}_{(n-4)/2}.$$

Unfortunately, this formula hides a lot of detail and does not seem to be of much immediate use. It is not valid in the curved case.

Recall that, like Q, the Pfaffian is an n-form canonically associated to a Riemannian metric on an oriented manifold in even dimensions. It is defined as a complete contraction of $n/2$ copies of the Riemann tensor with two copies of the volume form. For example, in dimension four it is

$$E = \epsilon^{abpq}\epsilon^{cdrs}R_{abcd}R_{pqrs},$$

where ϵ_{abcd} is the volume form normalised, for example, so that

$$\epsilon^{abcd}\epsilon_{abcd} = 4! = 24.$$

Therefore, in four dimensions,

$$\begin{aligned} E &= 4R_{ab}{}^{ab}R_{cd}{}^{cd} - 16R_{ab}{}^{ac}R_{cd}{}^{bd} + 4R_{ab}{}^{cd}R_{cd}{}^{ab} \\ &= 4R^2 - 16R_b{}^cR_c{}^b + 4C_{abcd}C^{abcd} + 32\mathrm{P}^{ab}\mathrm{P}_{ab} + 16\mathrm{P}^2 \\ &= 144\mathrm{P}^2 - 16(4\mathrm{P}^{ab}\mathrm{P}_{ab} + 8\mathrm{P}^2) + 4C_{abcd}C^{abcd} + 32\mathrm{P}^{ab}\mathrm{P}_{ab} + 16\mathrm{P}^2 \\ &= 32\mathrm{P}^2 - 32\mathrm{P}^{ab}\mathrm{P}_{ab} + 4C_{abcd}C^{abcd}. \end{aligned}$$

The integral of the Pfaffian on a compact manifold is a multiple of the Euler characteristic. In dimension 4, for example,

$$\int_M E = 128\pi^2\,\chi(M).$$

Notice the simple relationship between Q and E in dimension 4:–

$$Q = \tfrac{1}{16}E - \tfrac{1}{4}C^{abcd}C_{abcd} - \Delta\mathrm{P}.$$

Of course, it follows from (3) that $\int_M Q$ is a conformal invariant. Also, in the conformally flat case, it follows from a theorem of Branson, Gilkey, and Pohjanpelto that Q must be a multiple of the Pfaffian plus a divergence. However, the link between Q and the Pfaffian is extremely mysterious.

Conundrum: Find a direct link between Q and the Pfaffian in the conformally flat case. Prove directly that $\int_M Q$ is a topological invariant in this case.

Conundrum: Is it true that, on a general Riemannian manifold, Q may be written as a multiple of the Pfaffian plus a local conformal invariant plus a divergence?

Recall the conventions for Weyl structures as in the appendix. In particular, a metric in the conformal class determines a 1-form α_a. In fact, a Weyl structure may be regarded as a pair (g_{ab}, α_a) subject to equivalence under the simultaneous replacements

$$g_{ab} \mapsto \widehat{g}_{ab} = \Omega^2 g_{ab} \quad \text{and} \quad \alpha_a \mapsto \widehat{\alpha}_a = \alpha_a + \Upsilon_a \quad \text{where } \Upsilon_a = \nabla_a \Omega.$$

A Riemannian structure induces a Weyl structure by taking the equivalence class with $\alpha_a = 0$ but not all Weyl structures arise in this way. A Weyl structure gives rise to a conformal structure by discarding α_a. We may ask how Q-curvature is related to Weyl structures. From the transformation property (3), it follows that Q may be defined for a Weyl

structure as follows. Since Q is a Riemannian invariant, the differential operator P is necessarily of the form $f \mapsto S^a \nabla_a f$ for some Riemannian invariant linear differential operator from 1-forms to n-forms. Now, if $[g_{ab}, \alpha_a]$ is a Weyl structure, choose a representative metric g_{ab} and consider the n-form

$$Q - S^a \alpha_a,$$

where Q is the Riemannian Q-curvature associated to g_{ab} and α_a is the 1-form associated to g_{ab}. If $\widehat{g}_{ab} = \Omega^2 g_{ab}$, then

$$\begin{aligned} \widehat{Q} - \widehat{S}^a \widehat{\alpha}_a &= Q + S^a \Upsilon_a - \widehat{S}^a \alpha_a - \widehat{S}^a \Upsilon_a \\ &= Q + P \log \Omega - \widehat{S}^a \alpha_a - \widehat{P} \log \Omega \\ &= Q - \widehat{S}^a \alpha_a. \end{aligned} \tag{14}$$

In dimension 4 we can proceed further as follows. From (2) we see that

$$S^a \alpha_a = \nabla_b \left[\nabla^b \nabla^a + 2R^{ab} - \tfrac{2}{3} R g^{ab} \right] \alpha_a = \nabla_b \left[\nabla^b \nabla^a + 4\mathrm{P}^{ab} - 2\mathrm{P} g^{ab} \right] \alpha_a$$

and so we may calculate

$$\widehat{S}^a \alpha_a = S^a \alpha_a + 4\nabla^a (\Upsilon^b \nabla_{[a} \alpha_{b]}).$$

In combination with (14) we obtain

$$\widehat{Q} - \widehat{S}^a \widehat{\alpha}_a = Q - S^a \alpha_a - 4\nabla^a (\Upsilon^b \nabla_{[a} \alpha_{b]}).$$

However,

$$\widehat{\nabla}^a (\widehat{\alpha}^b \widehat{\nabla}_{[a} \widehat{\alpha}_{b]}) = \nabla^a (\widehat{\alpha}^b \nabla_{[a} \alpha_{b]}) = \nabla^a (\alpha^b \nabla_{[a} \alpha_{b]}) + \nabla^a (\Upsilon^b \nabla_{[a} \alpha_{b]})$$

and, therefore,

$$\mathbf{Q} = Q - S^a \alpha_a + 4\nabla^a (\alpha^b \nabla_{[a} \alpha_{b]}) \tag{15}$$

is an invariant of the Weyl structure that agrees with Q when the Weyl structure arises from a Riemannian structure.

Conundrum: Can we find such a $\mathbf{Q}$ in general even dimensions? Presumably, this would restrict the choice of Riemannian Q.

Though $\mathbf{Q}$ given by (15) is an invariant of the Weyl structure, it is not manifestly so. Better is to rewrite it as follows. Using conventions from the appendix, we may write the Schouten tensor (18) of the Weyl structure in terms of the Schouten tensor of a representative metric g_{ab}:–

$$\mathbf{P}_{ab} = \mathrm{P}_{ab} + \nabla_a\alpha_b + \alpha_a\alpha_b - \tfrac{1}{2}\alpha^c\alpha_c g_{ab}.$$

In particular,

$$\begin{array}{rcl}
\mathbf{P} & = & \mathrm{P} + \nabla^a\alpha_a - \alpha^a\alpha_a \\
\mathbf{P}^{ab}\mathbf{P}_{ab} & = & \mathrm{P}^{ab}\mathrm{P}_{ab} + (\nabla^a\alpha^b)(\nabla_a\alpha_b) + (\alpha^a\alpha_a)^2 + 2\mathrm{P}^{ab}\nabla_a\alpha_b \\
& & \quad + 2\mathrm{P}^{ab}\alpha_a\alpha_b - \mathrm{P}\alpha^a\alpha_a + 2(\nabla^a\alpha^b)\alpha_a\alpha_b - (\nabla^a\alpha_a)\alpha^b\alpha_b \\
D^aD_a\mathbf{P} & = & D^a(\nabla_a\mathbf{P} + 2\alpha_a\mathbf{P}) = \nabla^a(\nabla_a\mathbf{P} + 2\alpha_a\mathbf{P}) \\
& = & \Delta\mathbf{P} + 2(\nabla^a\alpha_a)\mathbf{P} + 2\alpha_a\nabla^a\mathbf{P} \\
& = & \Delta\mathrm{P} + \Delta\nabla^b\alpha_b - 2(\Delta\alpha^b)\alpha_b - 2(\nabla^a\alpha^b)(\nabla_a\alpha_b) \\
& & \quad + 2(\nabla^a\alpha_a)\mathrm{P} + 2(\nabla^a\alpha_a)\nabla^b\alpha_b - 2(\nabla^a\alpha_a)\alpha^b\alpha_b \\
& & \quad\quad + 2\alpha_a\nabla^a\mathrm{P} + 2\alpha_a\nabla^a\nabla^b\alpha_b - 4\alpha_a(\nabla^a\alpha^b)\alpha_b \\
\mathbf{P}^2 & = & \mathrm{P}^2 + (\nabla^a\alpha_a)^2 + (\alpha^a\alpha_a)^2 \\
& & \quad + 2\mathrm{P}\nabla^a\alpha_a - 2\mathrm{P}\alpha^a\alpha_a - 2(\nabla^a\alpha_a)(\alpha^b\alpha_b).
\end{array}$$

Therefore, recalling the formula (39) for Q in dimension 4,

$$\begin{aligned}
Q = 2\mathbf{P}^2 &- 2\mathbf{P}^{ab}\mathbf{P}_{ab} - D^aD_a\mathbf{P} \\
&+ 4\mathrm{P}^{ab}\nabla_a\alpha_b + 4\mathrm{P}^{ab}\alpha_a\alpha_b + \Delta\nabla^b\alpha_b - 2(\Delta\alpha^b)\alpha_b \\
&\quad + 2\alpha_a\nabla^a\mathrm{P} + 2\alpha_a\nabla^a\nabla^b\alpha_b - 2\mathrm{P}\nabla^a\alpha_a + 2\mathrm{P}\alpha^a\alpha_a
\end{aligned}$$

whence, from (15),

$$\begin{aligned}
\mathbf{Q} &= 2\mathbf{P}^2 - 2\mathbf{P}^{ab}\mathbf{P}_{ab} - D^aD_a\mathbf{P} \\
&\quad + 4\mathrm{P}^{ab}\nabla_a\alpha_b + 4\mathrm{P}^{ab}\alpha_a\alpha_b + \Delta\nabla^b\alpha_b - 2(\Delta\alpha^b)\alpha_b \\
&\quad\quad + 2\alpha_a\nabla^a\mathrm{P} + 2\alpha_a\nabla^a\nabla^b\alpha_b - 2\mathrm{P}\nabla^a\alpha_a + 2\mathrm{P}\alpha^a\alpha_a \\
&\quad\quad\quad - \nabla_b\left[\nabla^b\nabla^a + 4\mathrm{P}^{ab} - 2\mathrm{P}g^{ab}\right]\alpha_a + 4\nabla^a(\alpha^b\nabla_{[a}\alpha_{b]}) \\
&= 2\mathbf{P}^2 - 2\mathbf{P}^{ab}\mathbf{P}_{ab} - D^aD_a\mathbf{P} \\
&\quad + 4\mathrm{P}^{ab}\alpha_a\alpha_b + 2\alpha_a(\nabla^a\nabla^b - \nabla^b\nabla^a)\alpha_b + 2\mathrm{P}\alpha^a\alpha_a \\
&\quad\quad + 2(\nabla^a\alpha^b)\nabla_a\alpha_b - 2(\nabla^a\alpha^b)\nabla_b\alpha_a \\
&= 2\mathbf{P}^2 - 2\mathbf{P}^{ab}\mathbf{P}_{ab} - D^aD_a\mathbf{P} + 2(\nabla^a\alpha^b)\nabla_a\alpha_b - 2(\nabla^a\alpha^b)\nabla_b\alpha_a.
\end{aligned}$$

However,

$$2(\nabla^a\alpha^b)\nabla_a\alpha_b - 2(\nabla^a\alpha^b)\nabla_b\alpha_a = 4(\nabla^{[a}\alpha^{b]})\nabla_{[a}\alpha_{b]} = 4\mathbf{P}^{ab}\mathbf{P}_{[ab]}$$

and so

$$\mathbf{Q} = 2\mathbf{P}^2 - 2\mathbf{P}^{ab}\mathbf{P}_{ba} - D^aD_a\mathbf{P}$$

a manifest invariant of the Weyl structure, as required.

Conundrum: Did we really need to go through this detailed calculation? What are the implications, if any, for the operator S : 1-forms $\to$ 4-forms?

Conundrum: Can we characterise the Riemannian Q by sufficiently many properties? Do Weyl structures help in this regard?

Tom Branson has suggested that, for two metrics g and $\widehat{g} = \Omega^2 g$ in the same conformal class on a compact manifold M, one should consider the quantity

$$\mathcal{H}[\widehat{g}, g] = \int_M (\log\Omega)(\widehat{Q} + Q).$$

That it is a cocycle,

$$\mathcal{H}[\widehat{\widehat{g}},\widehat{g}]+\mathcal{H}[\widehat{g},g]=\mathcal{H}[\widehat{\widehat{g}},g],$$

is easily seen to be equivalent to the GJMS operators P being self-adjoint.

Conundrum: Are there any deeper properties of Branson's cocycle $\mathcal{H}[\widehat{g},g]$?

One possible rôle for Q is in a curvature prescription problem:–

Conundrum: On a given manifold M, can one find a metric with specified Q?

One can also ask this question within a given conformal class or within the realm of conformally flat metrics though, of course, if M is compact, then $\int_M Q$ must be as specified by the conformal class and the topology of M. There is also the question of uniqueness:–

Conundrum: When does Q determine the metric up to constant rescaling within a given conformal class?

Since we know how Q changes under conformal rescaling (3), this question is equivalent to

Conundrum: When does the equation $Pf=0$ have only constant solutions?

On a compact manifold in two dimensions this is always true: harmonic functions are constant. In four dimensions, though there are conditions under which $Pf=0$ has only constant solutions, there are also counterexamples, even on conformally flat manifolds. The following counterexample is due to Michael Singer and the first author. Consider the metric in local coördinates

$$\frac{dx^2+dy^2}{(x^2+y^2+1)^2}+\frac{ds^2+dt^2}{(s^2+t^2-1)^2}.$$

It is easily verified that it is conformally flat, scalar flat, and has

$$R_{ab}=4\frac{dx^2+dy^2}{(x^2+y^2+1)^2}-4\frac{ds^2+dt^2}{(s^2+t^2-1)^2}.$$

From (2) we see that if f is a function of (x,y) alone, then $Pf=L(L+8)f$, where L is the Laplacian for the two-dimensional metric

$$\frac{dx^2+dy^2}{(x^2+y^2+1)^2}.$$

More specifically, in these local coördinates

$$L=(x^2+y^2+1)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

It is easily verified that $L+8$ annihilates the following functions:–

$$\frac{x}{x^2+y^2+1},\qquad\frac{y}{x^2+y^2+1},\qquad\frac{x^2+y^2-1}{x^2+y^2+1}.$$

In fact, (x,y) are stereographic coördinates on the sphere and these three functions extend to the sphere to span the spherical harmonics of minimal non-zero energy. On then other hand, the metric

$$\frac{ds^2+dt^2}{(s^2+t^2-1)^2}$$

is the hyperbolic metric on the disc. We conclude that the Paneitz operator has at least a 4-dimensional kernel on $S^2\times H^2$. The same conclusion applies to $S^2\times\Sigma$ where Σ is any

Riemann surface of genus ≥ 2 equipped with constant curvature metric as a quotient of H^2. (In fact, the dimension in this case is exactly 4.)

APPENDIX: Curvature Conventions

Firstly, our conventions for conformal weight. A density f of conformal weight w may be identified as a function for any metric in the conformal class. At the risk of confusion, we shall also write this function as f. If however, our choice of metric g_{ab} is replaced by a conformally equivalent $\widehat{g}_{ab} = \Omega^2 g_{ab}$, then the function f is replaced by $\widehat{f} = \Omega^w f$. Quantities that are not conformally invariant can still have a conformal weight with respect to constant rescalings. For example, the scalar curvature has weight -2 in this respect. Explicit conformal rescalings are generally suppressed.

The Riemann curvature is defined by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\omega_c = R_{abc}{}^d \omega_d.$$

The Ricci and scalar curvatures are

$$R_{ac} = R_{abc}{}^b \quad \text{and} \quad R = R^a{}_a,$$

respectively. The Schouten tensor is

$$\mathrm{P}_{ab} = \frac{1}{n-2}\left(R_{ab} - \frac{R}{2(n-1)} g_{ab}\right)$$

and transforms under conformal rescaling by

$$\widehat{\mathrm{P}}_{ab} = \mathrm{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \tfrac{1}{2} g_{ab} \Upsilon^c \Upsilon_c. \tag{16}$$

In particular, if $\widehat{\eta}_{ab} = \Omega^2 \eta_{ab}$ are two flat metrics, then

$$\nabla_a \Upsilon_b = \Upsilon_a \Upsilon_b - \tfrac{1}{2} g_{ab} \Upsilon^c \Upsilon_c,$$

a tensor version of the Riccati equation. When $n = 2$, the Schouten tensor itself is not defined but its trace is well-defined:–

$$\mathrm{P} = \tfrac{1}{2} R \qquad \widehat{\mathrm{P}} = \mathrm{P} - \nabla^a \Upsilon_a = \mathrm{P} - \Delta \log \Omega \tag{17}$$

and so, if $\widehat{\eta}_{ab} = \Omega^2 \eta_{ab}$ are two flat metrics, then $\Delta \log \Omega = 0$.

A Weyl structure is a conformal structure together with a choice of torsion-free connection D_α preserving the conformal structure. In other words, if we choose a metric g_{ab} in the conformal class, then

$$D_a g_{bc} = 2\alpha_a g_{bc},$$

determining a smooth 1-form α_a. Conversely, α_a determines D_a:

$$D_a \phi_b = \nabla_a \phi_b + \alpha_a \phi_b + \alpha_b \phi_a - \alpha^c \phi_c g_{ab},$$

where ∇_a is the Levi-Civita connection for the metric g_{ab}. Let W_{ab} denote the Ricci curvature of the connection D_a:–

$$(D_a D_b - D_b D_a) V^c = W_{ab}{}^c{}_d V^d \qquad W_{ab} = W_{ca}{}^c{}_b.$$

We may compute these curvatures in terms of α_a and ∇_a, for a chosen metric in the conformal class:–

$$\begin{aligned} D_aD_bV^c \;=\;& \nabla_a(\nabla_bV^c-\alpha_bV^c+\alpha^cV_b-\alpha_dV^d\delta_b{}^c)\\ &+\alpha_a(\nabla_bV^c-\alpha_bV^c+\alpha^cV_b-\alpha_dV^d\delta_b{}^c)\\ &+\alpha_b(\nabla_aV^c-\alpha_aV^c+\alpha^cV_a-\alpha_dV^d\delta_a{}^c)\\ &+\alpha^e(\nabla_eV^c-\alpha_eV^c+\alpha^cV_e-\alpha_dV^d\delta_e{}^c)g_{ab}\\ &-\alpha_a(\nabla_bV^c-\alpha_bV^c+\alpha^cV_b-\alpha_dV^d\delta_b{}^c)\\ &+\alpha^c(\nabla_bV_a-\alpha_bV_a+\alpha_aV_b-\alpha_dV^dg_{ba})\\ &-\alpha_e(\nabla_bV^e-\alpha_bV^e+\alpha^eV_b-\alpha_dV^d\delta_b{}^e)\delta_a{}^c \end{aligned}$$

so

$$\begin{aligned} (D_aD_b-D_bD_a)V^c \;=\;& (\nabla_a\nabla_b-\nabla_b\nabla_a)V^c\\ &-(\nabla_a\alpha_b)V^c+(\nabla_a\alpha^c)V_b-(\nabla_a\alpha_d)V^d\delta_b{}^c\\ &+(\nabla_b\alpha_a)V^c-(\nabla_b\alpha^c)V_a+(\nabla_b\alpha_d)V^d\delta_a{}^c\\ &+\alpha^c\alpha_aV_b+\alpha_b\delta_a{}^c\alpha_eV^e-\alpha_e\alpha^e\delta_a{}^cV_b\\ &-\alpha^c\alpha_bV_a-\alpha_a\delta_b{}^c\alpha_eV^e+\alpha_e\alpha^e\delta_b{}^cV_a \end{aligned}$$

whence

$$\begin{aligned} W_{ab}{}^c{}_d \;=\;& R_{ab}{}^c{}_d-2\delta^c{}_d\nabla_{[a}\alpha_{b]}-2g_{d[a}\nabla_{b]}\alpha^c+2\delta_{[a}{}^c\nabla_{b]}\alpha_d\\ &+2\alpha^c\alpha_{[a}g_{b]d}+2\delta_{[a}{}^c\alpha_{b]}\alpha_d-2\alpha_e\alpha^e\delta_{[a}{}^cg_{b]d} \end{aligned}$$

and

$$W_{ab}=R_{ab}+(n-1)\nabla_a\alpha_b-\nabla_b\alpha_a+g_{ab}\nabla^c\alpha_c+(n-2)\alpha_a\alpha_b-(n-2)\alpha^c\alpha_cg_{ab}$$

whose trace is

$$W=R+2(n-1)\nabla^c\alpha_c-(n-1)(n-2)\alpha^c\alpha_c.$$

Therefore,

$$\frac{1}{n-2}\left(W_{ab}-\frac{W}{2(n-1)}g_{ab}\right)=\mathrm{P}_{ab}+\nabla_a\alpha_b+\alpha_a\alpha_b-\tfrac{1}{2}\alpha^c\alpha_cg_{ab}+\tfrac{2}{n-2}\nabla_{[a}\alpha_{b]}.$$

If two Weyl structures have the same underlying conformal structure, then we may, without loss of generality, represent them as (g_{ab},α_a) and $(g_{ab},\alpha_a-\Upsilon_a)$ for the same metric g_{ab} and an arbitrary 1-form Υ_a. If we write hatted quantities to denote those computed with respect to $(g_{ab},\alpha_a-\Upsilon_a)$, then

for

$$\mathbf{P}_{ab}=\frac{1}{n-2}\left(W_{ab}-\frac{2}{n}W_{[ab]}-\frac{W}{2(n-1)}g_{ab}\right)\tag{18}$$

we have the convenient transformation law

$$\widehat{\mathbf{P}}_{ab}=\mathbf{P}_{ab}-D_a\Upsilon_b+\Upsilon_a\Upsilon_b-\tfrac{1}{2}g_{ab}\Upsilon^c\Upsilon_c.\tag{19}$$

Chapter B: Origins, applications, and generalizations of the Q-curvature

by Tom Branson and Rod Gover

Curvature prescription

Everything below will take place in the setting of Riemannian manifolds (or Riemannian conformal manifolds) of even dimension n. Of course many statements will also be true for odd-dimensional manifolds and/or pseudo-Riemannian (conformal) manifolds, but our main intent is to make this blurb readable. There will be no reference list here, though there are plans to compile a separate reading list (of real papers) on the topic.

A touchstone in Differential Geometry is the *Yamabe equation*: for $n > 2$,

$$\left(\Delta + \frac{n-2}{4(n-1)} K\right) u = \frac{n-2}{4(n-1)} \hat{K} u^{\frac{n+2}{n-2}}. \tag{20}$$

This gives the conformal change law for the scalar curvature K. That is, if ω is a smooth function and

$$\hat{g} = e^{2\omega} g, \qquad u := e^{\frac{n-2}{2}\omega},$$

the $\hat{K}$ given by (20) is the scalar curvature of $\hat{g}$.

Exercise 1. Show that (20) implies the conformal change law for the *conformal Laplacian*

$$Y := \Delta + \frac{n-2}{4(n-1)} K,$$

namely

$$\hat{Y} = e^{-\frac{n+2}{2}\omega} Y e^{\frac{n-2}{2}\omega}.$$

Here, and in such formulas below, the function on the very right is to be viewed as a multiplication operator, so the relation really says that for all smooth functions f,

$$\hat{Y} f = e^{-\frac{n+2}{2}\omega} Y \left(e^{\frac{n-2}{2}\omega} f\right).$$

When $n = 2$, the equation governing the conformal change of K is qualitatively different from a PDE standpoint:

$$\Delta\omega + \frac{1}{2} K = \frac{1}{2} \hat{K} e^{2\omega}. \tag{21}$$

Like the Yamabe equation, this is quasilinear, but in contrast to Yamabe, it has an exponential (as opposed to power) nonlinearity, and it has an inhomogeneity (the $\frac{1}{2}K$ term). (21) is called the *Gauss curvature prescription equation*, the Gauss curvature in dimension 2 being $\frac{1}{2}K$.

There is a formal procedure of analytic continuation in dimension (which in fact can be made rigorous) that allows one to guess (or prove) (21) given (20). The Yamabe equation may be rewritten as

$$\Delta\left(e^{\frac{n-2}{2}\omega} - 1\right) + \frac{n-2}{4(n-1)} K e^{\frac{n-2}{2}\omega} = \frac{n-2}{4(n-1)} \hat{K} e^{\frac{n+2}{2}\omega}.$$

Note that we have slipped in an extra $-\Delta 1 = 0$ on the left. The advantage of this is that, as a power series in $\frac{n-2}{2}$, all terms in the equation begin at the first power. Dividing by $\frac{n-2}{2}$ and then evaluating at $n = 2$, we get (21).

Exercise 2 (following C.R. Graham). Make the dimensional continuation argument rigorous by looking at *stabilisations* of the manifold M, i.e. the $(n+p)$-dimensional manifolds $M \times T^p$, where T^p is the standard p-torus.

There is a generalisation of this whole picture to higher order, in which the role of the pair (Y, J) is played by a pair (P_m, Q_m) consisting of an operator and a local scalar invariant. The P_m are the celebrated Graham-Jenne-Mason-Sparling (GJMS) operators, which by construction have the following properties.

- P_m exists for m even and $m - n \notin 2\mathbb{Z}^+$.
- $P_m = \Delta^{m/2} + \text{LOT}$. (Here and below, LOT="lower order terms".)
- P_m is formally self-adjoint.

- P_m is conformally invariant in the sense that
$$\hat{P}_m = \exp\left(-\frac{n+m}{2}\omega\right) P_m \exp\left(\frac{n-m}{2}\omega\right).$$
- P_m has a polynomial expression in ∇ and R in which all coefficients are rational in the dimension n.
- P_m has the form
$$\delta S_m d + \frac{n-m}{2} Q_m,$$
where Q_m is a local scalar invariant, and S_m is an operator on 1-forms of the form
$$(d\delta)^{m/2-1} + \mathrm{LOT} \quad \text{or} \quad \Delta^{m/2-1} + \mathrm{LOT}.$$

In this last expression, d and δ are the usual de Rham operators and Δ is the form Laplacian $\delta d + d\delta$. Note that P_m is unable to detect changes in the $(d\delta)^{m/2-1}$ term of the principal part of S_m, sandwiched as it is between a δ and a d.

With these properties, conditions are right to generalise the Yamabe equation to
$$\left(P_m + \frac{n-m}{2} Q_m\right) u = \frac{n-m}{2} \hat{Q}_m u^{\frac{n+m}{n-m}}, \qquad m \notin \{n, n+2, n+4, \ldots\}, \tag{22}$$
where
$$u := e^{\frac{n-m}{2}\omega}.$$
Analytic continuation in dimension then yields the following analogue of the Gauss curvature prescription equation. If we denote P_n and Q_n simply by P and Q, then
$$\boxed{P\omega + Q = \hat{Q} e^{n\omega}.} \tag{23}$$
Though there is much more to be said about the Q-curvature, this is probably the central formula of the theory.

Exercise 3. Show that if we have a local invariant B satisfying a conformal change law like (23), $A\omega + B = \hat{B} e^{n\omega}$, with A a natural differential operator, then necessarily A is conformally invariant in the sense $\hat{A} = e^{-n\omega} A$.

The fact that P_m has an expression with rational dependence on the dimension is crucial to making the analytic continuation rigorous, whether one does it by stablilisation (generalising Exercise 2), or by an algebraic argument using $\mathbb{R}(n)$-linear combinations in a dimension-stable basis of invariants.

Very explicit formulas for P_m and Q_m are known up to $m = 8$. The $m = 4$ case, which was already being discussed in the early 1980's, is particularly appealing as a source of intuition, since the formulas there are still quite manageable. Let r be the Ricci tensor and let
$$\mathsf{P} := \frac{r - \mathsf{J} g}{n-2}.$$
The *Paneitz operator* is
$$P_4 := \delta S_4 d + \frac{n-4}{2} Q_4,$$
where
$$Q_4 := -2|\mathsf{P}|^2 + \frac{n}{2}\mathsf{J}^2 + \Delta \mathsf{J},$$
and
$$S_4 := d\delta + (n-2)\mathsf{J} - 4\mathsf{P}\,\cdot\,.$$

Here $\mathsf{P}\cdot$ is the natural action of a symmetric 2-tensor on 1-forms.

The transition from $m = 2$ to $m = 4$ already points up the fact that the (P_m, Q_m) are not uniquely determined: if C is the Weyl conformal curvature tensor, we could add a suitable multiple of $|C|^2$ to P_4 without destroying any of its defining properties. (The coefficient should be rational in n, should have a zero at $n = 4$, and should not have poles that create new "bad" dimensions.)

The $m = 4$ situation is also already big enough to show that the study of Q is not just a disguised study of the conformal properties of the Pfaffian Pff. One of the salient properties of the Pfaffian is that it can be written as a polynomial in R, without any explicit occurrences of ∇. For example, in dimension 4,

$$32\pi^2 \mathsf{Pff}_4 = |C|^2 - 8|\mathsf{P}|^2 + 8\mathsf{J}^2.$$

But the $\Delta\mathsf{J}$ term in Q_4 is an absolutely essential aspect, and it generalises: see Exercise 6 below.

All the conformal change laws we've mentioned are good in odd dimensions, and that for P_4 in dimension 3 shows that very strange nonlinearities can occur:

$$\left(\delta(d\delta + \mathsf{J} - 4\mathsf{P}\cdot)d - \frac{1}{2}Q_4\right)u = -\frac{1}{2}\hat{Q}_4 u^{-7},$$

where $u = e^{\omega/2}$.

Back to the general case, a celebrated property of Q is the conformal invariance of its integral on compact manifolds:

$$\int Q\, dv_g \ \text{ is conformally invariant.}$$

Indeed, since $dv_{\hat{g}} = e^{n\omega} dv_g$,

$$\int \hat{Q}\, dv_{\hat{g}} = \int (Q + P\omega) dv_g.$$

But $P\omega$ is an exact divergence, by the $\delta \cdots d$ property of P, and thus it integrates to 0, showing invariance. This has an immediate generalisation. If u is a smooth function,

$$\int \hat{Q} u\, dv_{\hat{g}} = \int (Q + P\omega) u\, dv_g.$$

Since P is formally self-adjoint, we may move it over to u in the very last term under the integral. If it happens that $u \in \mathcal{N}(P)$, there is no contribution from this term. Thus

$$u \in \mathcal{N}(P) \subset \mathcal{N}(d) \ \Rightarrow \ \int Qu\, dv_g \text{ is conformally invariant.}$$

Relativistic considerations

The Einstein equations are obtained by taking the *Einstein-Hilbert action* $\int K\, dv$ in dimension 4, and taking the *total metric variation*. This means we take a compactly supported symmetric tensor h and a curve of metrics g_ε with $(d/d\varepsilon)|_{\varepsilon=0} g_\varepsilon = h$, and compute that

$$(d/d\varepsilon)|_{\varepsilon=0} \left(\int_{\mathcal{K}} K\, dv\right) = \int \langle h, r - \tfrac{1}{2}Kg\rangle dv,$$

where $\mathcal{K}$ is any compact set containing supp(h). Here we may view $\langle\cdot,\cdot\rangle$ as the pairing of a covariant with a contravariant symmetric tensor, or as the metric (g_0) pairing of two covariant tensors.

Weyl relativity is one proposal for replacing the Einstein-Hilbert action with an action that is invariant under multiplication of the metric by a positive constant: under the variation above,

$$(d/d\varepsilon)|_{\varepsilon=0}\int_{\mathcal{K}}|C|^2dv=\int\langle h,\mathcal{B}\rangle dv,$$

where $\mathcal{B}$ is the *Bach tensor*, a trace-free symmetric 2-tensor with the property (when viewed as a contravariant tensor) that under the usual conformal variation,

$$\hat{\mathcal{B}}=e^{-2\omega}\mathcal{B}.$$

One aspect of $\mathcal{B}$ is that the nonlinear differential operator

$$g\mapsto\mathcal{B}_g, \tag{24}$$

carrying a metric to its Bach tensor, is fourth-order quasilinear. Its linearisation is an interesting fourth-order conformally invariant linear differential operator.

In attempting to generalise this to higher dimensions, it's clear that $\int \mathsf{Pff}$ won't help – its total metric variation is 0, since it's a topological invariant. Choices like $\int|C|^{n/2}$ for $n=8,12,\ldots$ are uninteresting because the linearisations of the analogues of the operators (24) have order lower than one might hope for – less than n. In fact these linearisations will even vanish when we vary at a conformally flat metric.

Coming to the rescue of the situation is Q:

$$(d/d\varepsilon)|_{\varepsilon=0}\int_{\mathcal{K}}Q\,dv=\int\langle h,\mathcal{A}\rangle dv,$$

where $\mathcal{A}$ is the *Fefferman-Graham tensor*, a symmetric contravariant 2-tensor with the conformal invariance law $\hat{\mathcal{A}}=e^{(2-n)\omega}\mathcal{A}$. The linearisation D of the map $g\mapsto\mathcal{A}_g$ has order n. The operator D is conformally invariant in the sense that $\hat{D}=e^{-(n+2)\omega}De^{-2\omega}$ when acting on trace-free symmetric contravariant 2-tensors.

Exercise 4. Show that if $\int S\,dv$ is conformally invariant, then its total metric variation tensor $\mathcal{C}$ is conformally invariant. Show that if T is any conformally invariant tensor, then the linearisation of the map $g\mapsto T_g$ is conformally invariant on trace-free perturbations of g (and 0 on pure trace perturbations).

Quantum considerations

Let A be a natural differential operator with positive definite leading symbol, and suppose A is a positive power of a conformally invariant operator. For example, A could be one of the GJMS operators, or it could be the square of the Dirac operator. Then in dimensions 2,4,6, and conjecturally in higher even dimensions,

$$-\log\frac{\det\hat{A}}{\det A}=\alpha\left\{\frac{1}{2}\int\omega P\omega\,dv+\int\omega Q\,dv\right\}+\int\left(\overline{F\,dv}-F\,dv\right)+\mathcal{H}, \tag{25}$$

where α is a constant, F is a local scalar invariant, and $\mathcal{H}$ is a term depending on the null space of A. In particular, if the conformally invariant condition $\mathcal{N}(A)=0$ is satisfied, then $\mathcal{H}=0$. The determinant involved is the zeta-regularised *functional determinant* of a positively elliptic operator.

Such formulas are the finite variational formulas corresponding to *Polyakov formulas*, which are infinitesimal variational formulas for the determinant; these take the form

$$(-\log\det A)^\bullet = \int \omega(L+h) \tag{26}$$

where L is a local invariant and h depends on the null space of A, and the superscripted bullet denotes the variation $(d/d\varepsilon)|_{\varepsilon=0}$. Here the curve along which we vary can be any curve of conformal metrics with $g^\bullet = 2\omega g$; it is particularly convenient to consider curves $g_\varepsilon = e^{2\varepsilon\omega}g_0$ for given g_0 and ω. As part of the package producing the Polyakov formula, one gets the *conformal index* property, that

$$\int L\,dv \ \text{ is conformally invariant.}$$

In fact, getting from (26) to (40) may be viewed as a process of finding *conformal primitives*. We say that a functional $\mathcal{F}$ on the conformal class $[g_0]$ is a conformal primitive for a local invariant L if $(d/d\varepsilon)|_{\varepsilon=0}\mathcal{F} = \int \omega L$. Of course this should happen at all possible choices of background metric g_0, and all directions of variation ω. This can be said in a more invariant way, following a suggestion of Mike Eastwood. Putting the "running" metric g and the background metric on the same footing in a two-metric functional $\mathcal{F}(\hat g, g)$ on the conformal class, we require of a conformal primitive that it be

- alternating and in the $\hat g$ and g arguments;
- cocyclic in the sense that

$$\mathcal{F}(\hat{\hat g}, g) = \mathcal{F}(\hat{\hat g}, \hat g) + \mathcal{F}(\hat g, g); \tag{27}$$

- having variation L in the sense above when varied in $\hat g$ for fixed g.

That the $\int(\overline{F\,dv} - F\,dv)$ term in (40) should be this way is obvious; for the term involving P and Q, it is a subtle point.

Some local invariants have other local invariants as conformal primitives. For example, since

$$\mathsf{J}^\bullet = -2\omega\mathsf{J} + \Delta\omega,$$

we have

$$(\mathsf{J}^{n/2})^\bullet = -n\omega\mathsf{J} + \frac{n}{2}\mathsf{J}^{n/2-1}\Delta\omega,$$

so

$$\left(\int \mathsf{J}^{n/2} dv\right)^\bullet = \frac{n}{2}\int \omega\Delta(\mathsf{J}^{n/2-1}).$$

This makes $2J^{n/2}/n$ a conformal primitive for $\Delta(\mathsf{J}^{n/2-1})$. Q is an example of a local invariant that does not have such a local conformal primitive.

In order to handle these objects more cleanly, let's view P and Q as being density-valued objects $\mathbf{P}$ and $\mathbf{Q}$, so that a "weight term" involving the conformal factor does not appear explicitly. In other words, replace Pf by $\mathbf{P}f = Pf\,dv_g$, and Q by $\mathbf{Q} = Q\,dv_g$. (For readers unfamiliar with densities, not much is lost conceptually in assuming our manifold is oriented and talking about n-forms instead of scalar densities.) Then

$$\hat{\mathbf{Q}} = \mathbf{Q} + \mathbf{P}\omega, \qquad \int \hat{\mathbf{Q}} = \int \mathbf{Q}. \tag{28}$$

Let's also take all the local invariants we consider to be density valued. A local primitive for $\mathbf{L}$ will be a functional $\mathcal{F}$ with $\mathcal{F}^\bullet = \int \omega \mathbf{L}$. The right side of (40) (ignoring the term $\mathcal{H}$, which may be shown to have a conformal primitive, or which may be eliminated by redefining the determinant) goes over to

$$\alpha \left\{ \frac{1}{2} \int \omega \mathbf{P} \omega + \int \omega \mathbf{Q} \right\} + \int \left(\hat{\mathbf{F}} - \mathbf{F} \right). \tag{29}$$

Recall that when we wrote this, we were thinking about a background metric g_0 and a perturbed metric g_ω. But there is an interesting way of rewriting the first term, as

$$\frac{1}{2} \alpha \int \omega (\mathbf{Q}_\omega + \mathbf{Q}_0). \tag{30}$$

To eliminate the appearance of a choice (of g_0) being made, we define $c(\hat{g}, g) := \omega$ for two conformally related metrics $\hat{g} = e^{2\omega} g$. The two-metric functional c is alternating and cocyclic as above. From this viewpoint, (30) is (α times) a two-point functional

$$\mathcal{G}(\hat{g}, g) := \frac{1}{2} \int c(\hat{g}, g)(\mathbf{Q}_{\hat{g}} + \mathbf{Q}_g),$$

and clearly

$$\mathcal{G}(\hat{g}, g) = -\mathcal{G}(g, \hat{g}).$$

The functional (29) is

$$\alpha \mathcal{G}(\hat{g}, g) + \int (\mathbf{F}_{\hat{g}} - \mathbf{F}_g). \tag{31}$$

Since the log-determinant functional will obviously satisfy the cocycle condition (27), and since the second functional in (31) satisfies such a condition, we expect $\mathcal{G}$ to behave similarly. One way to see that this expectation is fulfilled is to use the conformal primitive property: for fixed g and $\hat{g}$, with $\mathcal{G}$ for $\mathcal{F}$ above, the two sides of (27) have the same conformal variation (of $\hat{\hat{g}}$), and the same value at $\hat{\hat{g}} = g$.

Exercise 5. Show that if g_0, $g_\omega = e^{2\omega} g_0$, $g_\zeta = e^{2\omega} g_0$, and $g_\eta = e^{2\eta} g_0$ are 4 conformally related metrics, then

$$\int \omega (\mathbf{Q}_\zeta - \mathbf{Q}_\eta) + \int \zeta (\mathbf{Q}_\eta - \mathbf{Q}_\omega) + \int \eta (\mathbf{Q}_\omega - \mathbf{Q}_\zeta) = 0.$$

The following conjecture would be enough to prove the conjecture mentioned at the beginning of the section on the form of the determinant quotient.

Conjecture 1. If $\mathbf{S}$ is a natural n-form and $\int \mathbf{S}$ is conformally invariant, then

$$\mathbf{S} = \mathrm{const} \cdot \mathbf{Q} + \mathbf{L} + \mathbf{G},$$

where $\mathbf{L}$ is a local conformal invariant and $\mathbf{G}$ has a local conformal primitive. That is, there is a local invariant $\mathbf{F}$ for which the conformal variation of $\int \mathbf{F}$ is $\int \omega \mathbf{G}$.

The point of separating these 3 kinds of terms is that $\int \omega \mathbf{L}$ will have a very banal conformal primitive, namely itself, or (to write it in a way that makes the properties of a conformal primitive more apparent),

$$\frac{1}{2} \int c(\hat{g}, g)(\mathbf{L}_{\hat{g}} + \mathbf{L}_g).$$

$\mathbf{Q}$ has an interesting conformal primitive, as discussed above. $\mathbf{Q}$ is not uniquely defined, but the The $\mathbf{Q}$ in the statement of the conjecture could be anyone's favorite version of $\mathbf{Q}$. In fact, $\mathbf{Q}$ is well-defined up to addition of an $\mathbf{L}$.

Then there are the following related conjectures:

Conjecture 2. Any $\mathbf{S}$ as above may be written

$$\mathrm{const} \cdot \mathbf{Q} + \mathbf{L} + \mathbf{V},$$

where V is an exact divergence.

Conjecture 3. Any $\mathbf{S}$ as above may be written

$$\mathrm{const} \cdot \mathsf{Pff} + \mathbf{L} + \mathbf{V}.$$

There are at least 2 filtrations of the local invariants of this type that should be relevant. First, any invariant can be written as a sum of monomial expressions in R and ∇ with $k_\nabla + 2k_R = n$, where k_∇ (resp. k_R) is the number of occurrences of ∇ (resp. R) in the monomial. If an invariant $\mathbf{T}$ can be written with $k_\nabla \le p$ for each monomial term, let's say $\mathbf{T} \in \mathcal{P}_p$. Then

$$\mathcal{P}_0 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_{n-2}, \qquad\qquad \mathcal{P}_{\mathrm{odd}} = 0.$$

Pff is in the most elite space, $\mathcal{P}_0$. The exact divergences inject into $\mathcal{P}_2/\mathcal{P}_0$.

Exercise 6. Use the conformal change law for $\mathbf{Q}$ to show that the class of $\mathbf{Q}$ in $\mathcal{P}_{n-2}/\mathcal{P}_{n-4}$ is nontrivial, and agrees with the class of $(\Delta^{(n-2)/2}\mathsf{J})dv_g$. This establishes that Pff and $\mathbf{Q}$ are "at opposite ends" of the $\mathcal{P}$-filtration.

The other filtration is by the degree of the conformal change law. If

$$\hat{\mathbf{T}} = \mathbf{T} + \mathbf{T}_1(d\omega) + \cdots + \mathbf{T}_n(d\omega),$$

with $\mathbf{T}_i$ of homogeneity i with respect to scalar multiples of ω, then say $\mathbf{T} \in \mathcal{L}_p$ if $\mathbf{T}_q = 0$ for $q > p$. Then local conformal invariants are in $\mathcal{L}_0$, and $\mathbf{Q}$ is in $\mathcal{L}_1$. Pff on the other hand does not look so great in this filtration.

Other routes to Q and its variants

There is an alternative definition of Q which avoids dimensional continuation. We write $\mathcal{E}$ for the space of smooth functions, $\mathcal{E}^1$ for space of smooth 1-forms and define the special section

$$I^g := \begin{pmatrix} 2-n \\ 0 \\ \mathsf{J} \end{pmatrix}$$

of the direct sum bundle $\mathcal{E} \oplus \mathcal{E}^1 \oplus \mathcal{E}$. Let us first set the dimension to be 4, simply present the some results and then explain how this works. Then we get

$$\Box I^g = \begin{pmatrix} 0 \\ 0 \\ Q_4 \end{pmatrix},$$

where $\Box$ is the coupled conformal Laplacian operator. More precisely $\Box = -\nabla^a\nabla_a + (n-2)K/(4n-4)$, which appears to be the usual formula for the conformal Laplacian (cf. Y above), but now ∇ is a connection which couples the usual metric connection with the connection

$$\nabla_a \begin{pmatrix} \sigma \\ \mu \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla\sigma - \mu \\ \nabla\mu + g\tau + \mathsf{P}\sigma \\ \nabla\tau - \mu \lrcorner \mathsf{P} \end{pmatrix}.$$

on the sum bundle $\mathcal{T} := \mathcal{E} \oplus \mathcal{E}^1 \oplus \mathcal{E}$. This bundle $\mathcal{T}$ is called the standard tractor bundle and this connection is usually termed the (normal conformal) tractor connection. It is equivalent to a principal bundle structure known as the (normal conformal) Cartan connection. For those who know about Cartan connections we can say that the tractor bundle and connection is an associated bundle and connection for the Cartan bundle. We have used a metric g to express these objects in terms of a Riemannian structure but in fact the bundle and connection are conformally invariant and so descend to well defined structures on a conformal manifold. In fact, to be more accurate, the decomposition of the standard tractor bundle $\mathcal{T}$ is really $\mathcal{T} = \mathcal{E}[1] \oplus \mathcal{E}^1 \otimes \mathcal{E}[1] \oplus \mathcal{E}[-1]$ where $\mathcal{E}[w]$ indicates the space of conformal densities of weight w. The field I^g is a section of $\mathcal{T} \otimes \mathcal{E}[-1]$. In this section and the next we are allowing tensors and tractor fields to be density valued, to simplify the notation, but partially suppressing the details of weights involved.

This construction generalises. In each even dimension n there is a conformally invariant differential operator $\Box_{n-2}$ so that for any metric g we have

$$\Box_{n-2} I^g = \begin{pmatrix} 0 \\ 0 \\ Q_n \end{pmatrix}. \tag{32}$$

Here I^g is as above, while $\Box_{n-2}$ has the form $\Delta^{n/2-1}$ + LOT. The tractor field I^g is not conformally invariant, but it does have an interesting conformal transformation. If $\hat{g}$ is a metric related to g conformally according to $\hat{g} = e^{2\omega} g$ (ω a smooth function) then

$$I^{\hat{g}} = I^g + D\omega, \tag{33}$$

where D is a well known second order conformally invariant linear differential operator known as the tractor D operator. From this and (41) it follows that the Q-curvature $\hat{Q}_n$, for $\hat{g}$, differs from Q_n by a linear conformally invariant operator acting on ω. In fact it follows easily from the definition of $\Box_{n-2}$ that

$$\Box_{n-2} D\omega = \begin{pmatrix} 0 \\ 0 \\ P_n\omega \end{pmatrix}$$

where P_n is the GJMS operator of order n. So we have recovered the now famous property $\hat{Q}_n = Q_n + P_n\omega$ (cf. (28)).

As a final comment on the above story we should clarify the origins of the tractor field I^g defined and used above. For those who are familiar with tractors a more enlightening alternative definition is

$$I^g_A := -\frac{1}{n} D^A \sigma^{-1} D_{AB} \sigma.$$

Here D^A is the tractor D operator and D_{AB} is the so-called fundamental D operator. $0 \neq \sigma \in \Gamma\mathcal{E}[1]$ is the conformal scale corresponding to the metric g. The point is that these operators are both conformally invariant and under $g \mapsto \hat{g} = e^{2\omega} g$ we have $\sigma \mapsto e^{\omega}\sigma$. Since D_{AB} satisfies a Leibniz rule and $\sigma^{-1} D_{AB}\sigma$ is a "logarithmic derivative" the conformal transformation law of I^g is no surprise.

The main results above are derived via the ambient metric construction of Fefferman and Graham. Explaining this construction would be a significant detour at this point. Suffice to say that this construction geometrically associates to an n-dimensional conformal manifold M an $(n+2)$-dimensional pseudo-Riemannian manifold $\tilde{M}$. The GJMS operators $P_{2\ell}$ arise from powers of the Laplacian $\tilde{\Delta}^\ell$, of $\tilde{M}$, acting on suitably homogeneous functions. The operators $\Box_{2\ell}$ arise in a similar way from $\tilde{\Delta}^\ell$ on appropriately homogeneous sections of the tangent bundle $T\tilde{M}$. Such homogeneous sections correspond to tractor fields on the conformal manifold M. The results above are given by an easy calculation on the ambient manifold. Thus we can take (41) as a definition of the Q-curvature; it is simply the natural scalar field that turns up on the right hand side.

While this definition avoids dimensional continuation, there is still the issue of getting a formula for Q_n. There is an effective algorithm for re-expressing the ambient results in terms of tractors which then expand easily into formulae in terms of the underlying Riemannian curvature and its covariant derivatives. This solves the problem for small n. For example

$$Q_6 = 8\mathsf{P}_{ij|k}\mathsf{P}^{ij}{}_{|}{}^{k} + 16\mathsf{P}_{ij}\mathsf{P}^{ij}{}_{|k}{}^{k} - 32\mathsf{P}_{ij}\mathsf{P}^{i}{}_{k}\mathsf{P}^{jk} - 16\mathsf{P}_{ij}\mathsf{P}^{ij}\mathsf{J} +$$
$$8\mathsf{J}^3 - 8\mathsf{J}_{|k}{}^{k}\mathsf{J} + \mathsf{J}_{|j}{}^{j}{}_{k}{}^{k} + 16\mathsf{P}_{ij}\mathsf{P}_{kl}\mathsf{C}^{ikjl}.$$

While such formulae shed some light on the nature of the Q-curvature it would clearly be ideal to give a general formula or simple inductive formula. From the angle discussed here the missing information is a general formula for the operators $\Box_{n-2}$.

Problem 1: Give general formulae or inductive formulae for the operators $\Box_{2\ell}$.

This seems to be a difficult problem. In another direction there is another exercise to which we already have some answers. One of the features of the Q-curvature is that it "transforms by a linear operator" within a conformal class. More precisely, it is an example of a natural Riemannian tensor-density field with a transformation law

$$N^{\hat{g}} = N^g + L\omega, \tag{34}$$

L being some universal linear differential operator. (Here ω has the usual meaning; $\hat{g} = e^{2\omega}g$.)

Problem 2: Construct other natural tensor-densities which transform according to (43). (Note that any solution yields a conformally invariant natural operator L.)

From the transformation law for I^g above, we can evidently manufacture solutions to this problem. We have observed already that I^g is a section of the bundle $\mathcal{T}[-1] := \mathcal{T}\otimes\mathcal{E}[-1]$. If P is any scalar (or rather density) valued natural conformally invariant differential operator which acts on $\mathcal{T}[-1]$ then P can act on I^g, and PI^g has a conformal transformation of the the form (43). Using the calculus naturally associated to tractor bundles (or equally effectively, using the ambient metric) it is in fact a simple matter to write down examples, and the possibilities increase with dimension. This is most interesting when the resulting scalar field gives a possible modification to the original Q-curvature. For those familiar with densities this means that P should take values in densities of weight $-n$; this is the weight at which densities that can be integrated on a conformal manifold. For example, in any dimension we may take P to be $\iota(D)|C|^2$ where $\iota(D)$ indicates a contracted action of the tractor D operator and the square of the Weyl curvature is here viewed as a multiplication operator. In dimension 6 this takes values in $\mathcal{E}[-6]$ and we have

$$\iota(D)|C|^2 I^g = 4\Delta|C|^2$$

and

$$\iota(D)|C|^2 I^{\hat{g}} = 4\Delta|C|^2 - 16\delta|C|^2 d\omega.$$

Note that in this example the conformally invariant "L-operator" $\delta|C|^2 d$ is formally self-adjoint. So for any constant α, $Q_6 + \alpha\iota(D)|C|^2 I^g$ is another scalar field with almost the same properties as Q_6. It is not so closely related to the GJMS operator P_6, but it is related instead to a modification of P_6 by $\delta|C|^2 d$. It is clear that solutions to Problem 2 have a role to play in the problem of characterising the Q-curvature and the GJMS operators.

A generalisation: maps like Q

So far we have viewed the Q-curvature as a natural scalar field. It turns out that if instead we view it as an operator then it fits naturally into a bigger picture. To simplify matters suppose we are working with a compact, oriented, but not necessarily connected, manifold of even dimension n. We fix n and so omit n in the notation for Q. We can view Q as a multiplication operator from the closed 0-forms $\mathcal{C}^0$ (i.e. the locally constant functions) into the space of n-forms $\mathcal{E}^n$ (which we identify with $\mathcal{E}[-n]$ via the conformal Hodge $\star$). With the observations above we have the following properties:

A. $Q : \mathcal{C}^0 \to \mathcal{E}^n$ is not conformally invariant but $\hat{Q} = Q + P_n\omega$, where P_n is a formally self-adjoint operator from 0-forms to n-forms. P_n has the form dMd which implies the next properties.
B. $Q : \mathcal{C}^0 \to H^n(M)$ is conformally invariant and non-trivial in general.
C. If $c \in \mathcal{C}^0$ and $u \in \mathcal{N}(P_n)$ then $\int uQc$ is conformally invariant.
D. (See the discussion immediately below.) In each choice of metric $Q : \mathcal{E}^0 \to \mathcal{E}[-n]$ is formally self-adjoint.
E. (See the discussion immediately below.) $Q1$ is the Q-curvature.

The last properties are trivial since the operator is multiplication by a scalar field and by definition $Q1 = Q$. However we should note that we can add to Q any differential operator that annihilates constants, and properties 1–3 will be unaffected. Property 4 is suggesting that if we do that, then we should insist that the result is formally self-adjoint.

The idea now is to look for analogous operators on other forms. We write $\mathcal{C}^k$ for the space of closed k-forms. Consider the operator

$$M^g = d\delta + 2\mathsf{J} - 4\mathsf{P}\sharp : \mathcal{E}^{n/2-1} \to \mathcal{E}^{n/2-1}[-2]. \tag{35}$$

(Note that by the conformal Hodge star $\mathcal{E}^{n/2-1}[-2] \cong \mathcal{E}^{n/2+1}$, so we can also view this as an operator into $(n/2+1)$-forms.) Then:

Exercise 7. On $\mathcal{C}^{n/2-1}$ we have

$$M^{\hat{g}} = M^g + \beta\delta d\omega,$$

where β is some nonzero constant, $\hat{g} = e^{2\omega}g$, and in the display ω is viewed as a multiplication operator.

Note that the conformal variation term δd is the Maxwell operator and is formally self-adjoint. So M^g satisfies the analogue of property 1 above. The analogue of property 3 is an immediate consequence, i.e., $\int\langle u, Mc\rangle$ is conformally invariant where now c is a closed $(n/2-1)$-form and $u \in \mathcal{N}(\delta d)$ (so in fact by compactness u, c are both closed). Next observe, by inspection, that M^g is formally self-adjoint. So we have analogues for 1,3,4. There is also a bonus property, which is clear from the transformation law displayed:

$$\delta M^g d : \mathcal{E}^{n/2-2} \to \mathcal{E}^{n/2-2}[-4]$$

is a non-trivial conformally invariant operator. In dimension 4 this is the Paneitz operator.

So finally we need an analogue for property 2. It is clear that M^g is conformally invariant as a map $\mathcal{C}^{n/2-1} \to \mathcal{E}^{n/2-1}[-2]/\mathcal{R}(\delta)$, so this is an analogue. But we can do more. There is no reason to suppose the image is co-closed. On the other hand note that δM^g is conformally invariant on $\mathcal{C}^{n/2-1}$ and so we have the following:

$$\begin{array}{l} M^g : \mathcal{H}^{n/2-1} \to H^{n/2+1}(M) \quad \text{with} \\ \mathcal{H}^{n/2-1} := \mathcal{N}(\delta M : \mathcal{C}^{n/2-1} \to \mathcal{E}^{n/2-2}[-4]) \end{array} \tag{36}$$

is conformally invariant. The space $\mathcal{H}^{n/2-1}$ may be viewed as the space of "conformal harmonics". Evidently $\dim(\mathcal{H}^{n/2-1})$ is not always the same as the Betti number $b^{n/2-1}$, but the elliptic coercivity of the pair $(d, \delta M)$ gives it a good chance of returning the Betti number off a set of conformal structures that is somehow small. One should also check that the map (36) is non-trivial.

Fact: Let $M = S^p \times S^q$, where $p = n/2 - 1$, $q = n/2 + 1$, with the standard Riemannian structure. Then $\phi \in \mathcal{H}^p$ if and only if ϕ is harmonic. Furthermore, the map (36) is non-trivial.

In some recent work the authors have used the ambient metric, and its relationship to tractors, to show that the above construction generalises along the following lines: There are operators $M^g_k : \mathcal{E}^k \to \mathcal{E}^{n-k}$ $(k \leq n/2 - 1)$, given by a uniform construction, with the following properties:

A. $M^g_k : \mathcal{C}^k \to \mathcal{E}^{n-k}$ has the conformal transformation law $M^{\hat{g}}_k = M^g_k + L_k\omega$, where L_k is a formally self-adjoint operator from k-forms to $(n-k)$-forms, and is a constant multiple of $dM^g_{k+1}d$.

B. $\mathcal{H}^k := \mathcal{N}(dM^g_k : \mathcal{C}^k \to \mathcal{E}^{n-k+1})$ is a conformally invariant subspace of $\mathcal{C}^k$ and $M_k : \mathcal{H}^k \to H^{n-k}(M)$ is conformally invariant. There are conformal manifolds on which M_k is non-trivial.

C. If $c \in \mathcal{C}^k$ and $u \in \mathcal{N}(L_k)$ then

$$\int \langle u, M^g_k c \rangle$$

is conformally invariant.

D. For each choice of metric g, $M^g_k : \mathcal{E}^k \to \mathcal{E}^{n-k}$ is formally self-adjoint.

E. $M^g_0 1$ is the Q-curvature.

From the uniqueness of the Maxwell operator at leading order (as a conformally invariant operator $\mathcal{E}^{n/2-1} \to \mathcal{E}^{n/2-1}[-2]$), and the explicit formula (44) it is clear $M^g_{n/2-1}$ is not the difference between any conformally invariant differential operator and a divergence (even as an operator on closed forms). A similar argument applies to the M_k generally. Thus, from the point of view that the Q-curvature is a non-conformally invariant object that in a deep sense cannot be made conformally invariant, but one which nevertheless determines a global conformal invariant, the operators M^g_k give a genuine generalisation of the Q-curvature to an operator on closed forms.

Problems k: There are analogues for the operators M_k^g of most of the conundrums and problems for the Q-curvature.

Chapter C: Open problems

Conformal Structure in Geometry, Analysis, and Physics

August 12 to 16, 2003 at the

American Institute of Mathematics, Palo Alto, California

I. Problems suggested by the participants

Thomas Branson. Anti-conformal perturbations.
Problem 1a: Given any functional of the metric that is well understood conformally, is there information that can arise going across conformal classes?

If the functional is the integral of a local invariant we can obtain information by computing its anti-conformal variation. If the functional is a nonlocal spectral invariant, like the functional determinant, then it is even a challenge to compute the anti-conformal deformation.

Problem 1b: How to obtain the information that arises going across conformal classes?

Problem 1c: Study variational problems arising from conformally invariant problems.

Michael Eastwood.
Problem 2: Find an explicit relation between Q and $\mathsf{Pff}(R)$ in the conformally flat case.

Problem 3: Is there a global ambient metric construction?

Problem 4: Can we explicitly write Q in dimension 6 uniquely as constant times $\mathsf{Pff}(R)$ plus local conformally invariant plus divergence?

Answer to problem 4: Robin Graham reports the answer to be YES.

Alice Chang. General problems in conformal geometry:
Problem 5a: How to decide which curvature invariants have a conformal primitive? For example on manifold M, we have $\Delta(J^{n/2-1})$ has $\frac{2}{n}J^{n/2}$ as a conformal primitive, i.e.

$$\left(\int \frac{2}{n} J^{n/2}\right)^{\bullet}(\omega) = \Delta(J^{n/2-1})$$

for all smooth function ω on M, see "Origins, applications and generalizations of the Q-curvature" by T. Branson and R. Gover. Available through http://www.aimath.org.

Problem 5b: What characterizes such curvature invariants?

A related problem is posed by T. Branson:
On M^n, Q curvature is a local invariant (of density weight $-n$) which does not have a conformal primitive. The local invariants that have conformal primitives form a vector subspace, say L', of the space of local invariants $\mathcal{L}$. Thus the quotient space $\mathcal{L}/\mathcal{L}'$ is the

space which measures "how many things" do not have a conformal primitive. There are also local conformal invariants, $\mathcal{L}''$ say.

Problem 6: Is $\mathcal{L}/(\mathcal{L}' + \mathcal{L}'')$ one-dimensional and generated by the class of Q?

Problem 7: On M^4, Gursky ("The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE." Comm. Math. Phys., 207(1):131-143, 1999.) proved that if $\mathrm{Sc} > 0$, and if $\int Q > 0$, then the Paneitz operator P_4 is positive with its kernel consisting of constants. The original proof given by Gursky depends on estimates of solution of some non-linear PDE. Can one also see this fact from the construction method of the general GJMS operators?

Claude LeBrun.
Problem 8: Explicitly expess the Gauss-Bonnet integrand as a sum of $\sigma_{n/2}(\mathrm{P})$ plus terms involving the Weyl curvature, and then use this to explicitly understand relationships between Q and topology.

Problem 9: Given a compact manifold of even dimension > 2, show that there exists a sequence of metrics such that $\int Q \to +\infty$.

Robin Graham
Problem 10: If $n \geq 4$ is even, is there a nonzero scalar conformal invariant of weight $-n$ which is expressible as a linear combination of complete contractions of the tensors $\nabla^l \mathrm{P}$, $l \geq 0$?

If the answer to this question is no, then the Q-curvature defined via the ambient metric construction is uniquely determined by its transformation law in terms of the GJMS operator P_n and the fact that it can be written just in terms of P and its derivatives. The answer is no if $n = 4$. It is worth pointing out that there are scalar conformal invariants of more negative weight which can be so expressed: the norm squared of the Bach tensor is of this form if $n = 4$, as is the norm squared of the ambient obstruction tensor in higher even dimensions.

Problem 11: If $n \geq 4$ is even, is the GJMS operator P_n the only natural differential operator with principal part $\Delta^{n/2}$ whose coefficients can be expressed purely in terms of the tensors $\nabla^l \mathrm{P}$, $l \geq 0$, and which is conformally invariant from $\mathcal{E}(0)$ to $\mathcal{E}(-n)$?

If the answer is yes, then this gives a characterization of the GJMS operator P_n. Combined with a negative answer to Problem 11, this would provide a unique specification of Q.

Rod Gover Alice Chang and Jie Qing have an order 3 operator P_3, on 3-manifolds (boundary of a 4-dimensional manifold, or embedded in a 4-dimensional manifold). There is a version of Q_3 associated to this P_3.

Problem 12: What sort of information is encoded by Q_3 and/or $\int Q_3$?

Helga Baum On a spin manifold (M, g) with spin bundle S, we have two conformally covariant operators. The Dirac operator D_g and the twistor operator P_g. If ∇^S represents the spin connection then,

$$\nabla^S : \Gamma(S) \to \Gamma(T^*M \otimes S) \cong \Gamma(S) \oplus \Gamma(T_w)$$

and we define $D_g = \mathrm{pr}_1 \nabla^S$ and $P_g = \mathrm{pr}_2 \nabla^S$, with pr_i the projection on the i-th factor. Let $h(g)$ be the dimension of $\ker(D_g)$ (harmonic spinors) and let $t(g)$ be the dimension of $\ker(P_g)$ (twistor spinors/conformal Killing spinors). Both numbers are conformal invariants. In case of Riemannian conformal structures these invariants are rather well studied. In the Lorentzian case much less is known.

Problem 13: Find all Lorentzian conformal structures $(M, [g])$ with $t(g) > 0$ or $h(g) > 0$.

Problem 14: How $t(g)$ and $h(g)$ relate to other conformal invariants?

Problem 15: Relate $t(g)$ to the holonomy of conformal Cartan connections.

Problem 16: Relate $h(g)$ to the dynamic of null geodesics.

Problem 17: Describe conformally flat Lorentzian manifolds with $h(g) > 0$ or $t(g) > 0$.

II. Problems extracted from the document "A Primer on Q-curvature" by M. Eastwood and J. Slovàck. [1]

In the conformally flat case, locally by setting $g_{ab} = \Omega^2 \eta_{ab}$ where η_{ab} is flat, then

$$Q = \Delta^{n/2} \log \Omega, \tag{37}$$

where Δ is the ordinary Laplacian in Euclidean space with η_{ab} as metric. For this construction of Q to be well-defined it is necessary that, if also $g_{ab} = \widehat{\Omega}^2 \widehat{\eta}_{ab}$, then

$$\Delta^{n/2} \log \Omega = \widehat{\Delta}^{n/2} \log \widehat{\Omega}.$$

This reduces to two facts:–

fact 1:: $\Delta^{n/2}$ is conformally invariant on flat space.
fact 2:: if g_{ab} is itself flat, then $\Delta^{n/2} \log \Omega = 0$.

The second of these is necessary in order that (37) be well-defined. There is a Lie algebraic proof of **fact 1**. It corresponds to the existence of a homomorphism between certain generalized Verma modules for $\mathfrak{so}(n+1, 1)$.

Problem 18: Deduce **fact 2** from **fact 1** or vice versa. Alternatively, construct a Lie algebraic proof of **fact 2**.

About a formula for Q, Eastwood and Slováck have deduce:

$$\begin{aligned} \Delta^2 \log \Omega \;=\; & -\widehat{\Delta}\widehat{\mathrm{P}} - (n-2)\widehat{\mathrm{P}}^{ab}\widehat{\mathrm{P}}_{ab} + 2\widehat{\mathrm{P}}^2 \\ & + 2(n-4)\Upsilon^a \widehat{\nabla}_a \widehat{\mathrm{P}} + 2(n-4)\Upsilon^a \Upsilon_a \widehat{\mathrm{P}} \\ & - (n-2)(n-4)\Upsilon^a \Upsilon^b \widehat{\mathrm{P}}_{ab} + \tfrac{1}{4}(n-2)(n-4)\Upsilon^a \Upsilon_a \Upsilon^b \Upsilon_b. \end{aligned} \tag{38}$$

[1] This section is the recompilation of the conundra in that document. Refer to the original for more details.

Though it is only guaranteed that this formula is valid in the conformally flat case, in fact it agrees with the general expression in dimension 4,

$$Q = 2\mathrm{P}^2 - 2\mathrm{P}^{ab}\mathrm{P}_{ab} - \Delta\mathrm{P}. \tag{39}$$

It is possible, by further differentiating (38), to obtain a formula for $\Delta^k \log\Omega$ expressed in terms of complete contractions of $\widehat{\mathrm{P}}_{ab}$, its hatted derivatives, and Υ_a. With increasing k, this gets rapidly out of hand. Moreover, it is only guaranteed to give Q in the conformally flat case. Indeed, when $n = 6$ this naive derivation of Q fails for a general metric.

Problem 19: Find a formula for Q in the conformally flat case. Show that the procedure outlined by Eastwood and Slováck produces a formula for Q.

In the conformally flat case, it follows from a theorem of Branson, Gilkey, and Pohjanpelto that Q must be a multiple of the Pfaffian plus a divergence.

Problem 20: Find a direct link between Q and the Pfaffian in the conformally flat case. Prove directly that $\int_M Q$ is a topological invariant in this case.

Problem 21: Is it true that, on a general Riemannian manifold, Q may be written as a multiple of the Pfaffian plus a local conformal invariant plus a divergence?

See Problem 4 for the 6 dimensional case. Also, T. Branson has appointed that if it is true that any local invariant L of density weight $-n$ has the form

$$\text{constant}_L \mathsf{Pff} + \text{divergence}_L + (\text{local conformal invariant})_L$$

where L signals the dependence on L then, in this decomposition for Q, we have $\text{constant}_Q \neq 0$. In fact we know constant_Q exactly, since we know (the constant values of) Q and Pff on the sphere.

How is Q-curvature related to Weyl structures? Q may be defined for a Weyl structure as follows. Since Q is a Riemannian invariant, the differential operator P is necessarily of the form $f \mapsto S^a\nabla_a f$ for some Riemannian invariant linear differential operator from 1-forms to n-forms. Now, if $[g_{ab}, \alpha_a]$ is a Weyl structure, choose a representative metric g_{ab} and consider the n-form

$$Q - S^a\alpha_a,$$

where Q is the Riemannian Q-curvature associated to g_{ab} and α_a is the 1-form associated to g_{ab}. If $\widehat{g}_{ab} = \Omega^2 g_{ab}$, then

$$\widehat{Q} - \widehat{S}^a\widehat{\alpha}_a = Q - \widehat{S}^a\alpha_a.$$

In dimension 4, Eastwood and Slováck have appointed that

$$Q - S^a\alpha_a + 4\nabla^a(\alpha^b\nabla_{[a}\alpha_{b]})$$

is an invariant of the Weyl structure that agrees with Q when the Weyl structure arises from a Riemannian structure.

Problem 22: Can we find such a $\mathbf{Q}$ in general even dimensions? Presumably, this would restrict the choice of Riemannian Q.

Though $\mathbf{Q}$ is an invariant of the Weyl structure, it is not manifestly so. With a detailed calculation, Eastwood and Slov‘ack have shown that in dimension 4:

$$\mathbf{Q} = 2\mathbf{P}^2 - 2\mathbf{P}^{ab}\mathbf{P}_{ba} - D^a D_a \mathbf{P}$$

a manifest invariant of the Weyl structure.

Problem 23: Did we really need to go through that detailed calculation? What are the implications, if any, for the operator S : 1-forms $\to$ 4-forms?

Problem 24 a: Can we characterise the Riemannian Q by sufficiently many properties?

Problem 24 b: Do Weyl structures help in this regard?

Tom Branson has suggested that, for two metrics g and $\widehat{g} = \Omega^2 g$ in the same conformal class on a compact manifold M, one should consider the quantity

$$\mathcal{H}[\widehat{g}, g] = \int_M (\log \Omega)(\widehat{Q} + Q).$$

That it is a cocycle,

$$\mathcal{H}[\widehat{\widehat{g}}, \widehat{g}] + \mathcal{H}[\widehat{g}, g] = \mathcal{H}[\widehat{\widehat{g}}, g],$$

is easily seen to be equivalent to the GJMS operators P being self-adjoint.

Problem 25: Are there any deeper properties of Branson's cocycle $\mathcal{H}[\widehat{g}, g]$?

One possible rôle for Q is in a curvature prescription problem:

Problem 26: On a given manifold M, can one find a metric with specified Q?

One can also ask this question within a given conformal class or within the realm of conformally flat metrics though, of course, if M is compact, then $\int_M Q$ must be as specified by the conformal class and the topology of M.

Problem 27: When does Q determine the metric up to constant rescaling within a given conformal class?

Since we know how Q changes under conformal rescaling:

$$\widehat{Q} = Q + P \log \Omega,$$

where P is a linear differential operator from functions to n-forms whose symbol is $\Delta^{n/2}$ this question is equivalent to

Problem 28: When does the equation $Pf = 0$ have only constant solutions?

On a compact manifold in two dimensions this is always true: harmonic functions are constant. In four dimensions, though there are conditions under which $Pf = 0$ has only constant solutions, there are also counterexamples, even on conformally flat manifolds.

III. Problems extracted from the document "Origins, applications, and generalizations of the Q-curvature" by T. Branson and R. Gover. [2]

Let A be a natural differential operator with positive definite leading symbol, and suppose A is a positive power of a conformally invariant operator. For example, A could be one of the GJMS operators, or it could be the square of the Dirac operator. Then in dimensions 2,4,6,

$$-\log \frac{\det \hat{A}}{\det A} = \alpha \left\{ \frac{1}{2} \int \omega P \omega \, dv + \int \omega Q \, dv \right\} + \int \left(\overline{F \, dv} - F \, dv \right) + \mathcal{H}, \tag{40}$$

where α is a constant, F is a local scalar invariant, and $\mathcal{H}$ is a term depending on the null space of A. In particular, if the conformally invariant condition $\mathcal{N}(A) = 0$ is satisfied, then $\mathcal{H} = 0$. The determinant involved is the zeta-regularized *functional determinant* of a positively elliptic operator.

Problem 29: [3] Is (40) true in higher even dimensions?

The following conjecture would be enough to answer the previous problem.

Problem 30: If $\mathbf{S}$ is a natural n-form and $\int \mathbf{S}$ is conformally invariant, then

$$\mathbf{S} = \text{const} \cdot \mathbf{Q} + \mathbf{L} + \mathbf{G},$$

where $\mathbf{L}$ is a local conformal invariant and $\mathbf{G}$ has a local conformal primitive. That is, there is a local invariant $\mathbf{F}$ for which the conformal variation of $\int \mathbf{F}$ is $\int \omega \mathbf{G}$.

Problem 31: Is it possible to write any $\mathbf{S}$, as in Problem 30, in the form

$$\text{const} \cdot \mathbf{Q} + \mathbf{L} + \mathbf{V},$$

where V is an exact divergence?

Problem 32: Is it possible to write any $\mathbf{S}$, as in Problem 30, in the form

$$\text{const} \cdot \mathsf{Pff} + \mathbf{L} + \mathbf{V}?$$

Other routes to Q. There is an alternative definition of Q which avoids dimensional continuation. Let $\mathcal{E}$ be the space of smooth functions, let $\mathcal{E}^1$ be space of smooth 1-forms and define the special section

$$I^g := \begin{pmatrix} 2-n \\ 0 \\ J \end{pmatrix}$$

of the direct sum bundle $\mathcal{E} \oplus \mathcal{E}^1 \oplus \mathcal{E}$. In dimension 4:

$$\Box I^g = \begin{pmatrix} 0 \\ 0 \\ Q_4 \end{pmatrix},$$

[2] This section is the recompilation of the problems and conjectures in that document. We strongly suggest its lecture for a better understanding of the following problems.

[3] Problems 29 – 32 are actually conjectures that T. Branson and R Gover address in their document.

where

$$\Box = -\nabla^a \nabla_a + (n-2)K/(4n-4),$$

which appears to be the usual formula for the conformal Laplacian, but now ∇ is a connection which couples the usual metric connection with the connection

$$\nabla_a \begin{pmatrix} \sigma \\ \mu \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla\sigma - \mu \\ \nabla\mu + g\tau + V\sigma \\ \nabla\tau - \mu \lrcorner \mathrm{P} \end{pmatrix}$$

on the sum bundle $\mathcal{T} := \mathcal{E} \oplus \mathcal{E}^1 \oplus \mathcal{E}$. In any even dimension n, there is a conformally invariant differential operator $\Box_{n-2}$ so that for any metric g :

$$\Box_{n-2} I^g = \begin{pmatrix} 0 \\ 0 \\ Q_n \end{pmatrix}. \tag{41}$$

Here I^g is as above, while $\Box_{n-2}$ has the form $\Delta^{n/2-1} + \mathrm{lot}$ (with lot = "lower order terms"). If $\hat{g}$ is a metric related to g conformally according to $\hat{g} = e^{2\omega} g$ (ω a smooth function) then

$$I^{\hat{g}} = I^g + D\omega, \tag{42}$$

where D is a well known second order conformally invariant linear differential operator (the tractor D operator). From this and (41) it follows that the Q-curvature $\hat{Q}_n$, for $\hat{g}$, differs from Q_n by a linear conformally invariant operator acting on ω. In fact

$$\Box_{n-2} D\omega = \begin{pmatrix} 0 \\ 0 \\ P_n\omega \end{pmatrix}$$

where P_n is the GJMS operator of order n, recovering the property $\hat{Q}_n = Q_n + P_n\omega$.

While this definition avoids dimensional continuation, there is still the issue of getting a formula for Q_n. There is an effective algorithm for re-expressing the ambient results in terms of tractors which then expand easily into formulae in terms of the underlying Riemannian curvature and its covariant derivatives, solving the problem for small n.

Problem 33: Give general formulae or inductive formulae for the operators $\Box_{2\ell}$.

In another direction there is another exercise to which already are some answers. One of the features of the Q-curvature is that it "transforms by a linear operator" within a conformal class. More precisely, it is an example of a natural Riemannian tensor-density field with a transformation law

$$N^{\hat{g}} = N^g + L\omega, \tag{43}$$

L being some universal linear differential operator. (Here ω has the usual meaning; $\hat{g} = e^{2\omega} g$.)

Problem 34: Construct other natural tensor-densities which transform according to (43). (Note that any solution yields a conformally invariant natural operator L.)

Solutions to Problem 34 have a role to play in the problem of characterizing the Q-curvature and the GJMS operators.

Generalizations of Q. In a compact, oriented, but not necessarily connected, manifold of even dimension n, Q can be seen as a multiplication operator from the closed 0-forms $\mathcal{C}^0$ (i.e. the locally constant functions) into the space of n-forms $\mathcal{E}^n$ (identified with $\mathcal{E}[-n]$ via the conformal Hodge $\star$). This operator has the following properties:

A. $Q : \mathcal{C}^0 \to \mathcal{E}^n$ is not conformally invariant but $\hat{Q} = Q + P_n\omega$, where P_n is a formally self-adjoint operator from 0-forms to n-forms. P_n has the form dMd which implies the next properties.
B. $Q : \mathcal{C}^0 \to H^n(M)$ is conformally invariant and non-trivial in general.
C. If $c \in \mathcal{C}^0$ and $u \in \mathcal{N}(P_n)$ then $\int uQc$ is conformally invariant.
D. In each choice of metric $Q : \mathcal{E}^0 \to \mathcal{E}[-n]$ is formally self-adjoint.
E. $Q1$ is the Q-curvature.

The idea now is to look for analogous operators on other forms. T. Branson and R. Gover (see math.DG/0309085) have used the ambient metric, and its relationship to tractors, to show that the previous generalizes along the following lines: There are operators $M_k^g : \mathcal{E}^k \to \mathcal{E}^{n-k}$ ($k \le n/2 - 1$), given by a uniform construction, with the following properties:

A. $M_k^g : \mathcal{C}^k \to \mathcal{E}^{n-k}$ has the conformal transformation law $M_k^{\hat{g}} = M_k^g + L_k\omega$, where L_k is a formally self-adjoint operator from k-forms to $(n-k)$-forms, and is a constant multiple of $dM_{k+1}^g d$. Here $\mathcal{C}^k$ is the space of closed k-forms,
B. $\mathcal{H}^k := \mathcal{N}(dM_k^g : \mathcal{C}^k \to \mathcal{E}^{n-k+1})$ is a conformally invariant subspace of $\mathcal{C}^k$ and $M_k : \mathcal{H}^k \to H^{n-k}(M)$ is conformally invariant. There are conformal manifolds on which M_k is non-trivial.
C. If $c \in \mathcal{C}^k$ and $u \in \mathcal{N}(L_k)$ then
$$\int \langle u, M_k^g c\rangle$$
is conformally invariant.
D. For each choice of metric g, $M_k^g : \mathcal{E}^k \to \mathcal{E}^{n-k}$ is formally self-adjoint.
E. $M_0^g 1$ is the Q-curvature.

From the uniqueness of the Maxwell operator at leading order (as a conformally invariant operator $\mathcal{E}^{n/2-1} \to \mathcal{E}^{n/2-1}[-2]$), and the explicit formula
$$M^g = d\delta + 2J - 4\mathrm{P}\sharp : \mathcal{E}^{n/2-1} \to \mathcal{E}^{n/2-1}[-2], \tag{44}$$
it is clear $M_{n/2-1}^g$ is not the difference between any conformally invariant differential operator and a divergence (even as an operator on closed forms). A similar argument applies to the M_k generally. Thus, from the point of view that the Q-curvature is a non-conformally invariant object that in a deep sense cannot be made conformally invariant, but one which nevertheless determines a global conformal invariant, the operators M_k^g give a genuine generalization of the Q-curvature to an operator on closed forms.

Problem 35: There are analogues for the operators M_k^g of most of the Problems in Sections II. and III. for the Q-curvature.

Chapter D: Reference list

A list of references related to the topic of the workshop is available at
http://www.und.nodak.edu/instruct/lapeters/bib