

AIM CHIP FIRING 2013: THE F -LATTICE

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ABSTRACT. We record some of the discussions of the F -lattice group at AIM Chip Firing 2013. Our goal was to understand that scaling limit of the Abelian sandpile on the F -lattice. The contents of the introduction was known prior to the conference, but the rest is new.

1. INTRODUCTION

The F -lattice is the directed graph $(\mathbb{Z}^2, E)$ such that each vertex x has exactly two outgoing edges depending on parity: $(x, x \pm (1, 0))$ when $x_1 + x_2$ is even and $(x, x \pm (0, 1))$ when $x_1 + x_2$ is odd.

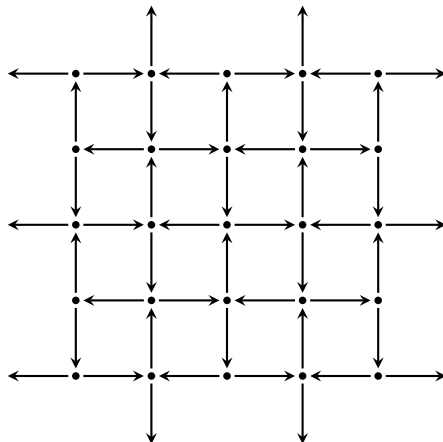


FIGURE 1.1. A 5×5 section of the F -lattice.

The F -lattice Laplacian of a function $u : \mathbb{Z}^2 \rightarrow \mathbb{R}$ is given by

$$\Delta_F u(x) := \sum_{(y,x) \in E} (u(y) - u(x)).$$

Note that the sum is over outgoing edges. Our goal is to prove the following conjecture, which would completely characterize the scaling limit of the Abelian sandpile on the F -lattice.

Conjecture 1.1. For each 2×2 real symmetric matrix A , the following are equivalent.

- (1) There is a function $u : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ such that

$$\Delta_F u(x) \leq 0 \quad \text{and} \quad u(x) = \frac{1}{2}x \cdot Ax + o(1 + |x|^2)$$

for all $x \in \mathbb{Z}^2$.

(2) The difference

$$\frac{1}{2} \begin{bmatrix} s-t & s+t \\ s+t & t-s \end{bmatrix} - A$$

is non-negative definite for some $s, t \in \mathbb{Z}$.

We call matrices satisfying the first condition *F-superharmonic* and denote the set of all such matrices Γ_F . Our conjecture is equivalent to the statement:

$$\partial\Gamma_F = \{M(a, b, \gamma_F(a, b)) : a, b \in \mathbb{R}^2\},$$

where $M : \mathbb{R}^3 \rightarrow S_2$ is the parameterization

$$M(a, b, c) := \frac{1}{2} \begin{bmatrix} c+a & b \\ b & c-a \end{bmatrix}$$

and $\gamma : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by

$$(1.1) \quad \gamma(x)_F := \max_{s, t \in \mathbb{Z}} -|x - (s - t, s + t)|.$$

That is, the boundary of Γ_F is a union of downward cones arranged on a grid:

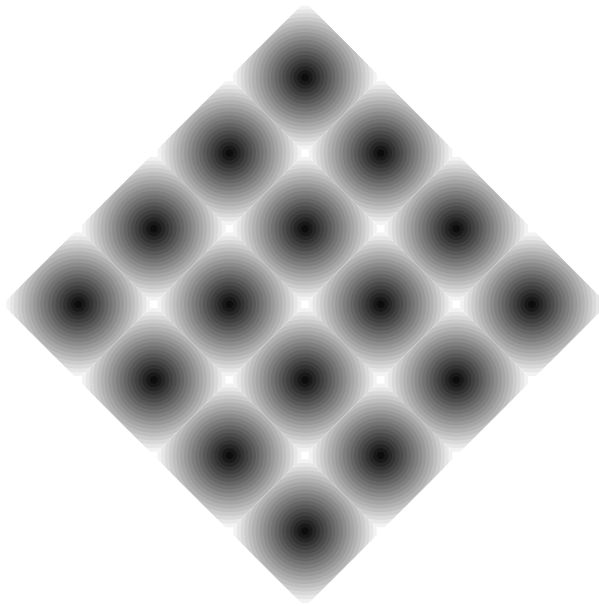


FIGURE 1.2. A heat plot of several periods of γ_F . Black is 0 and white is -1 .

The formula (1.1) and thus the picture Figure 1.2 has been checked (rigorously) by the computer for all $x \in \mathbb{R}^2$ such that $2^{10}x$ is an integer. Thus, we are confident the conjecture is true.

2. EXPLAINING THE PICTURE

Via some examples, we can explain why γ_F should be at least as periodic as Figure 1.2 suggests and (1.1) implies. One can verify that the functions

$$\begin{aligned} u_1(x) &:= \tfrac{1}{2}x_1(x_1 + 1) \\ u_2(x) &:= \lfloor \tfrac{1}{4}(x_1 + x_2)^2 \rfloor \\ u_3(x) &:= \lfloor \tfrac{1}{4}(x_1 - x_2)^2 \rfloor \end{aligned}$$

and matrices

$$\begin{aligned} A_1 &:= M(1, 0, 1) \\ A_2 &:= M(0, 1, 1) \\ A_3 &:= M(0, -1, 1) \end{aligned}$$

satisfy

$$\Delta u_i(x) = \tfrac{1}{2} - \tfrac{1}{2}(-1)^{x_1+x_2}$$

and

$$u_i(x) = \tfrac{1}{2}x \cdot A_i x + o(1 + |x|^2),$$

for all $x \in \mathbb{Z}^2$.

In the picture Figure 1.2, the matrices $-A_i$ are three of the four corners surrounding the peak at $M(0, 0, 0)$. Moreover, the differences $A_i - A_j$, which have corresponding F -harmonic $u_i - u_j$, generate the lattice of periodicity.

The miracle of the F -lattice is that the zero matrix, and its translations by $s(A_1 - A_4) + t(A_2 - A_3)$, are the only “efficient” F -superharmonic matrices. Everything else is below one of these “peaks” in the matrix order.

We also observe, using the symmetries of the lattice, that

$$\gamma_F(x_1, x_2) = \gamma_F(-x_1, x_2) = \gamma_F(x_2, x_1)$$

for all $x \in \mathbb{R}^2$.

3. TOWARDS A PROOF

Using the symmetries and the downward closure condition, it is enough to prove that

$$\gamma_F(t, 1-t) = -\sqrt{t^2 + (1-t)^2}$$

for $0 \leq t \leq 1$. In particular, we must show that

$$M(t, 1-t, -\sqrt{t^2 + (1-t)^2})$$

lies on the boundary of Γ_F for all $0 \leq t \leq 1$.

Our idea, following the proof in the square lattice case, is to enumerate the rational matrices A on this hyperbola, compute the corresponding F -superharmonic functions $u_A : \mathbb{Z}^2 \rightarrow \mathbb{Z}$, and hope that the periodic patterns Δu_A have some common structure.

Some of these patterns are displayed in Figure 3.1. We immediately observe that the determinant of the lattice is the denominator of the matrix and that later patterns appear to be constructed from earlier patterns. To be more concrete:

Question 3.1. Is there a Farey fraction type construction for the rational points on the hyperbola? Are the patterns enjoy an analogous construction?

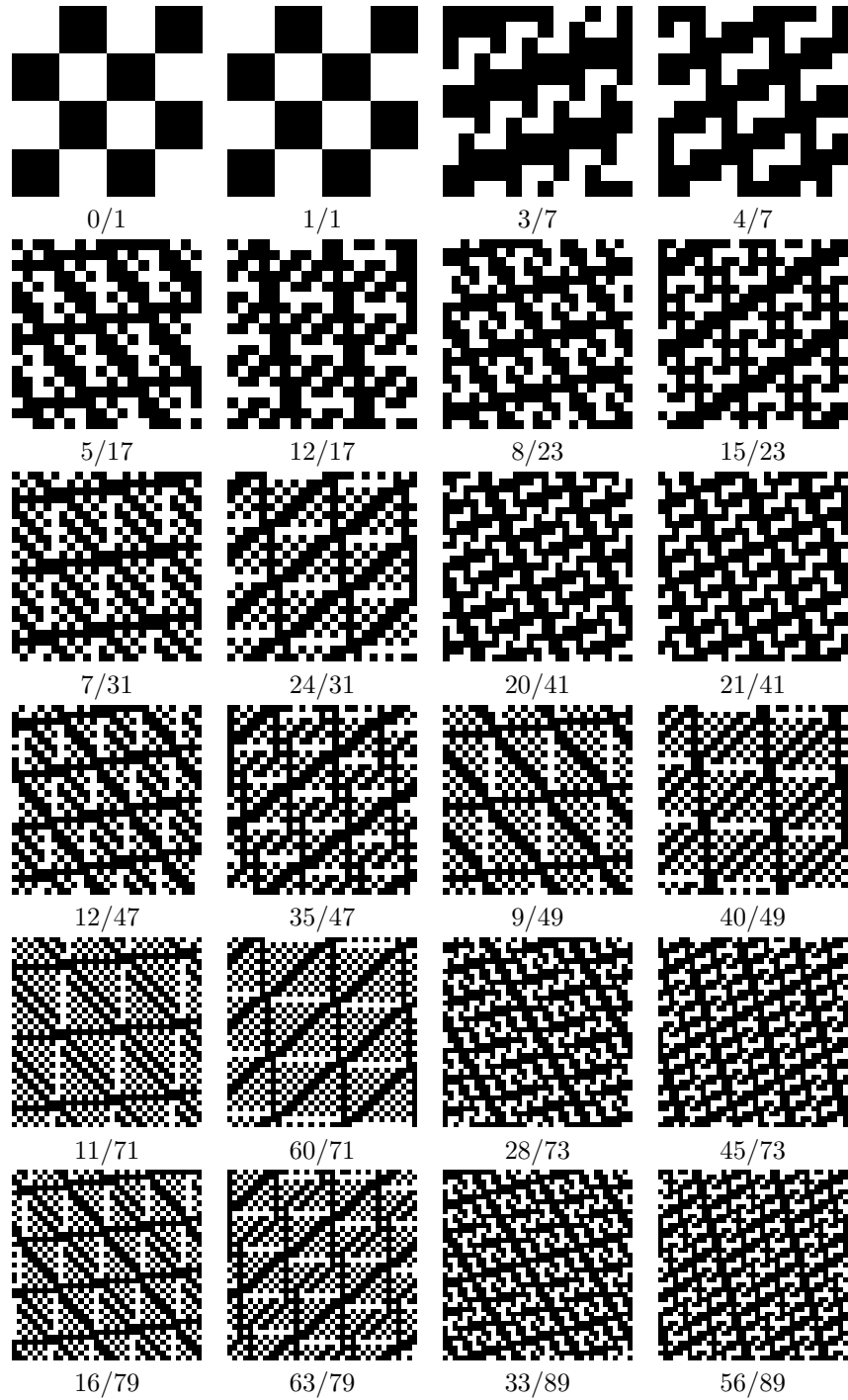


FIGURE 3.1. The first 24 patterns and rational t on the hyperbola.
Black is 0 and white is -1.