WARM-UP IN TYPE A: NONCROSSING, NONNESTING PARTITIONS AND ASSOCIAHEDRA

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1. The goal

These exercises are intended for the first "workshop" during the AIM conference on "Braid groups, clusters, and free probability", January 10-14, 2005. The goal is for the participants to work (in groups or individually) through some of the exercises in order to gain familiarity with some of the objects that will pervade the later discussions. The exercises are not intended to be challenging. It is hoped that people will choose to do the set of exercises pertaining to the objects with which they have the *least* prior familiarity.

An auxiliary goal of each set of exercises is to indicate how the various objects in type A may be generalized at least to all finite Weyl groups, and in some cases to all finite (real) reflection groups.

We have not given complete references or attributions for various facts. All references given are available at the conference web-site: `www.math.ucsb.edu/~mccammon/aim-conference`.

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2. Warm-up exercises on noncrossing partitions

Let $W = S_n$ denote the symmetric group on n letters. Let T denote the set of all transpositions $t = (i\,j)$ for $1 \leq i < j \leq n$, considered as a generating set[1] for W.

(a) Let $c(w)$ denote the number of cycles in the cycle decomposition of w, counting fixed points each as one cycle. Show that if $t = (i\,j)$ then

$$c(wt) = \begin{cases} c(w) - 1 & \text{if } i, j \text{ lie in different cycles of } w, \\ c(w) + 1 & \text{if } i, j \text{ lie in the same cycle of } w. \end{cases}$$

(b) Show that the *(absolute) length function*[2]

$$\ell_T(w) := \min\{\ell : w = t_1 t_2 \cdots t_\ell \text{ for some } t_i \in T\}$$

has these equivalent reformulations:

$$\ell_T(w) = n - c(w) = \operatorname{codim}_{\mathbb{R}}(V^w)$$

where $V = \mathbb{R}^n$ carries the defining representation of $W = S_n$ permuting coordinates, and V^w is the subspace of vectors fixed pointwise by w.

(c) Define a binary relation $<$ on W by taking the reflexive, transitive closure of the relation $w < wt$ for $w \in W, t \in T$ with $\ell_T(w) < \ell_T(wt)$. Show that $\leq$ is a partial order on W having the identity element e as its unique minimum element.

(d) Show that $u \leq v$ if and only if

$$\ell_T(u) + \ell_T(u^{-1}v) = \ell_T(v).$$

Show that ℓ_T gives a *rank function* for $\leq$ on W in the sense that every maximal totally ordered subset (*chain*) in the interval from e to w has length $\ell_T(w)$.

Recall the definition of the poset of noncrossing partitions $NC(n)$. It is a subposet of the refinement order Π_n on all partitions of the n-element set $[n] := \{1, 2, \ldots, n\}$, and consists of those partitions whose blocks have disjoint convex hulls when $[n]$ labels (in clockwise order) the vertices of a convex n-gon.

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[1] Warning: this is not a set of Coxeter generators for W, such as the set S of all adjacent transpositions.

[2] And, of course, this is not the usual Coxeter group length function $\ell_S(w)$.

(e) Let c be the n-cycle $(123\cdots n-1\,n)$ in W. Show that $w \leq c$ in the partial order on W if and only if the cycles of w form a noncrossing partition of $[n]$ in which each cycle is directed clockwise around the n-gon.

(f) Show that $NC(n) \cong [e,c]_<$ where $[e,c]_<$ denotes the interval from e to w in the partial order on W described above.

The next two exercises show that $NC(n)$, as well as each of its intervals, is self-dual as a poset. It is perhaps not crucial for a first pass through, but is important both for the discussion of Garside structures and for free probability.

Assume $u \leq w$ in the above partial order on W. Let v be an element in the interval $[u,w]_<$, so that $v = ua$ and $w = vb = uab$ for some a, b such that $\ell_T(u) + \ell_T(a) + \ell_T(b) = \ell_T(w)$. Define $\varphi(v) := u \cdot aba^{-1}$, so that $w = \varphi(v) \cdot a$.

(g) Show that φ maps $[u,w]_<$ into itself, bijectively, and reverses the order: if $u < v_1 < v_2 < w$ then $u < \varphi(v_1) < \varphi(v_2) < w$.

(h) In the framework of exercise (g), consider the situation when $u = e$ and $w = c$, so that $[u,w]_< \cong NC(n)$ by exercise (f). In this case φ corresponds under the isomorphism to an anti-automorphism ψ of the lattice $NC(n)$.

Prove that for a non-crossing partition $\pi \in NC(n)$, the partition $\psi(\pi)$ can be described as follows: subdivide the edges of the n-gon whose vertices are labelled $1, \ldots, n$, letting $1', \ldots, n'$ be the midpoints of the edges from n to 1, from 1 to $2, \ldots,$ from $n-1$ to n, respectively. Then $\psi(\pi)$ is the coarsest partition in the lattice $NC(\{1', \ldots, n'\})$ such that π and $\psi(\pi)$ together form a non-crossing partition of

$$\{1', 1, 2', 2, \ldots, n', n\}$$

with respect to this labelling of a $2n$-gon.

Remark 2.1. One can use the ideas in parts (a)-(f) to generalize the definition of $NC(n)$ from W of type A_{n-1} to any finite Coxeter system (W,S). Let V be the usual reflection representation of W of dimension $|S|$, so that the set T of W-conjugates of S are the reflections in this representation, and the (absolute) length $\ell_T(w)$ turns out to coincide with $\text{codim}_{\mathbb{R}}(V^w)$.

Let c be any *Coxeter element*, that is, a product of the elements of S in some order. Finiteness of W is well-known to imply that c is well-defined up to W-conjugacy. Finally, let $NC(W,S)$ be the poset $[e,c]_<$.

It turns out that $NC(W,S)$ is a graded, self-dual lattice of rank $|S|$, with rank function ℓ_T. Furthermore the map $w \mapsto V^w$ embeds $NC(W,S)$ as a subposet of Π_W, where Π_W is the poset (lattice) of all intersections of reflecting hyperplanes from W, ordered by reverse inclusion [2].

Case-by-case verifications have shown that $NC(W,S)$ has cardinality

$$\frac{1}{|W|}\prod_{i=1}^{r}(e_i+h+1)$$

where e_i are the *exponents* of W, and h is the *Coxeter number*.

3. Warm-up exercises on nonnesting partitions

Recall the definition of the poset of nonnesting partitions $NN(n)$. Given a partition π of the set $[n] := \{1,2,\ldots,n\}$, a *bump* of π is a pair (i,j) in the same block of π with no integers k having $i<k<j$ in the same block. Then $NN(n)$ is a subposet of the refinement order Π_n on all partitions of $[n]$, and consists of those partitions having no pair of bumps $(i,j) \neq (i',j')$ which are *nested*: $i \leq i' \leq j' \leq j$.

Consider the usual crystallographic root system Φ of type A_{n-1}, along with one of its usual choices of positive roots Φ^+:

$$\Phi := \{e_i - e_j : 1 \leq i \neq j \leq n\}$$
$$\Phi^+ := \{e_i - e_j : 1 \leq i < j \leq n\}$$

Let $\mathbb{N}\Phi^+$ denote the set of all nonnegative integral combinations of the positive roots.

(a) Show that a partition π of $[n]$ is the transitive closure of the relation determined by its collection of bumps, and hence that π is uniquely determined by its bumps. Show that the collection of bumps (i,j) of a partition π always corresponds to a linearly independent set of roots $e_i - e_j$ in Φ^+.

(b) Show that the relation on Φ^+ defined by

$$\alpha' \leq \alpha \text{ if } \alpha - \alpha' \in \mathbb{N}\Phi^+$$

defines a partial order on Φ^+. We will call this the *(positive) root poset*. Draw this poset for $n = 2,3,4,5$.

(c) Show that two bumps $(i,j),(i',j')$ are nested if and only if their corresponding roots $\alpha = e_i - e_j, \alpha' = e_{i'} - e_{j'}$ satisfy $\alpha' < \alpha$.

Consequently, the map that sends a nonnesting partition π to the roots corresponding to its bumps gives a bijection between $NN(n)$ and the set of all *antichains* (= collections of pairwise incomparable elements) in the positive root poset.

Recall that an *order filter* in a poset P is a subset $F \subseteq P$ with the property that $x \in F$ and $y > x$ in P implies $y \in F$.

(d) For any poset P show that the map sending an antichain A to the set $F := \{x \in P : x \geq a \text{ for some } a \in A\}$ gives a bijection between the antichains in P and the filters in P.

Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_n$ of all $n \times n$ matrices over $\mathbb{C}$, with a choice of *Borel subalgebra* $\mathfrak{b}$ consisting of all upper triangular matrices, and its *nilradical* $\mathfrak{n}$ consisting of all strictly upper triangular matrices. Recall the decomposition

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ for a root $\alpha = e_i - e_j$ is its *root subspace*, consisting of all matrices that contain at most one non-zero entry in row i and column j. Recall that an *ideal* $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ is a linear subspace satisfying $\mathfrak{a} \subseteq \mathfrak{g}$ satisfying $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$.

(e) Show that the ideals of $\mathfrak{g}$ contained in $\mathfrak{n}$ are exactly those of the form $\bigoplus_{\alpha \in F} \mathfrak{g}_\alpha$ where F is some filter in the positive root order on Φ^+. Explain how to biject such ideals with antichains in Φ^+ and nonnesting partitions.

Consider the *Shi arrangement* of hyperplanes in $V = \mathbb{R}^n$, namely the hyperplanes of the form $\langle x, \alpha \rangle = 0, 1$ as α ranges through Φ^+. Removing these hyperplanes from V leaves open connected components called *regions*. This arrangement is depicted for $n = 3$ in Figure 3, after modding out by the 1-dimensional subspace $x_1 = \cdots x_n$ which is parallel to all of the hyperplanes. Here the *positive cone* containing the regions where $\langle x, \alpha \rangle > 0$ is shown shaded.

(f) Given a region R of the Shi arrangement lying in the positive cone, let F be the collection of positive roots $\alpha \in \Phi^+$ satisfying $\langle x, \alpha \rangle > 1$

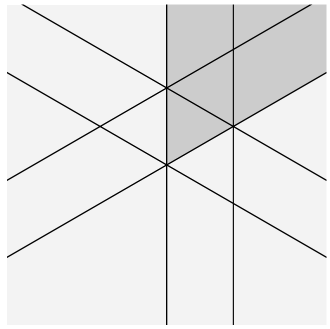

FIGURE 1. The Shi arrangement for $n = 3$ (or type A_2), with the positive cone shaded. Figure taken from [4]

for every $x \in R$. Show that R is a filter in the positive root order. Explain how this gives an injective map from the set of such regions to antichains in Φ^+ and nonnesting partitions. (This map turns out to be bijective, but this is not obvious.)

Remark 3.1. One can use this idea to generalize the definition of $NN(n)$ from the root system Φ of type A_{n-1} to any finite crystallographic root system Φ, along with a choice of positive roots Φ^+. Let $NN(\Phi)$ be the set of all antichains A in the positive root order on Φ^+, and order these antichains by reverse inclusion of their corresponding intersection subspace $\bigcap_{\alpha \in A} \alpha^\perp$.

It turns out that this map $A \mapsto \bigcap_{\alpha \in A} \alpha^\perp$ embeds $NN(\Phi)$ as a subposet of the poset (lattice) of intersections of the reflecting hyperplanes for Φ. It can also be shown that an antichain A of positive roots is always linearly independent, and hence the cardinality $|A|$ is the codimension of the corresponding intersection subspace$\bigcap_{\alpha \in A} \alpha^\perp$.

As above, one can biject these antichains A to filters F in the positive root order, and then to ideals $\mathfrak{a}$ in the nilradical $\mathfrak{n}$ of a Borel subalgebra $\mathfrak{b}$ for the associated semisimple Lie algebra $\mathfrak{g}$. One can also biject the regions of the Shi arrangement lying the positive (dominant) cone with such filters. The latter can be used to prove uniformly (see [1, 3]) that the number of antichains in the positive root poset obeys the formula

$$\frac{1}{|W|} \prod_{i=1}^{r} (e_i + h + 1)$$

where W is the Weyl group associated to Φ, the e_i are the *exponents* of W, and h is the *Coxeter number.*

4. Warm-up exercises on associahedra and clusters

Recall that the $(n-1)$-dimensional associahedron is a simple convex polytope whose vertices correspond to triangulations of an $(n+2)$-gon, and whose edges correspond to pairs of triangulations that differ by a *diagonal flip* within a quadrilateral common to both triangulations. The goal here will be to think about its dual polytope, the $(n-1)$-dimensional (simplicial) associahedron in a different way: we will define its boundary complex in a fairly natural way as an abstract simplicial complex.

Consider a vertex set V indexed by the interior diagonals of a convex $(n+2)$-gon P. Say that two such diagonals α, α' are *compatible* if they do not cross, that is, their interiors are disjoint. Let $G = (V, E)$ be the graph on vertex set V whose edge set E consists of pairs of compatible diagonals. Let Δ be the abstract simplicial complex on vertex V whose simplices are the subsets $F \subset V$ consisting of pairwise compatible diagonals[3].

(a) Explain how the simplices of Δ biject with *polygonal subdivisions* of P that introduce no new (interior) vertices, that is, decompositions of P into convex polygons whose vertices are a subset of the vertices of P. Explain how facets (=maximal simplices) of Δ biject with *triangulations* of P.

(b) Show that every facet of Δ has dimension $(n-2)$, that is, Δ is a pure $(n-2)$-dimensional complex.

(c) Show that two facets of Δ share an $(n-3)$-face if and only if their corresponding triangulations differ by a diagonal flip. Prove that Δ is an $(n-2)$-dimensional *pseudomanifold*: every $(n-3)$-face lies in exactly two facets, and for any two facets F, F' there exists is a gallery $F = F_0, F_1, \ldots, F_{r-1}, F_r = F'$ of facets in which F_i, F_{i+1} share an $(n-3)$-face for each $i = 0, 1, ..., r-1$.

In fact, since Δ happens to be the boundary complex of a simplicial $(n-1)$-dimensional polytope, it triangulates an $(n-2)$-sphere, but we won't prove that here. Rather, we wish to rephrase the compatibility graph $G = (V, E)$ (and hence also its clique complex Δ) in terms of the root system of type A_{n-1}.

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[3]Sometimes Δ is called the *clique complex* or *flag complex* associated with the graph G which is its 1-skeleton.

Consider the usual system of roots, positive roots, simple roots for type A_{n-1}:

$$\Phi := \{e_i - e_j : 1 \leq i \neq j \leq n\}$$
$$\Phi^+ := \{\alpha_{ij} = e_i - e_j : 1 \leq i < j \leq n\}$$
$$\Pi := \{\alpha_i = e_i - e_{i+1} : 1 \leq i \leq n-1\}$$

and define the *almost positive roots*

$$\Phi_{\geq -1} := \Phi^+ \sqcup (-\Pi).$$

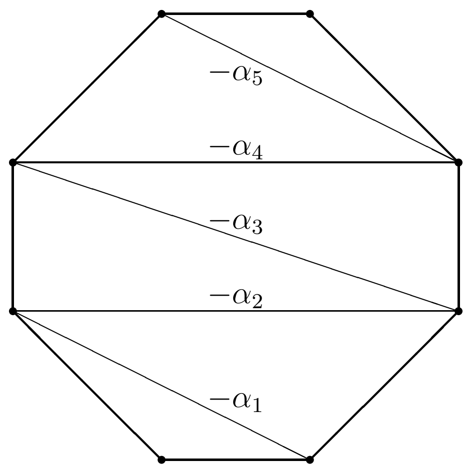

FIGURE 2. The "snake" of interior diagonals corresponding to the negative simple roots $-\Pi$ for $n = 6$ (or type A_5). Figure taken from [4].

(d) Show that $|V| = |\Phi_{\geq -1}| = \binom{n+1}{2} - 1$. Show that one can biject the set of interior diagonals V with $\Phi_{\geq -1}$ in the following way. Identify $-\Pi$ with the diagonals in the "snake" as depicted in Figure 2 below. Then for $1 \leq i < j \leq n$ identify the positive root

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$$

with the *unique* interior diagonal that crosses (i.e. is incompatible with) exactly the set of snake diagonals labelled by $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_{j-1}$.

(e) Show that for every interior diagonal in V, there exists at least one rotational symmetry of P carrying it into a snake diagonal.

(f) Note that two diagonals in V are compatible if and only if their images under any symmetry of P are compatible. Show that if a pair of diagonals correspond to almost positive roots α, α' with $\alpha = -\alpha_i \in -\Pi$

(i.e. α is in the snake), then the pair is compatible if and only if the unique expansion of α' in terms of the simple roots α_j has zero coefficient on α_i.

Remark 4.1. One can use this idea to generalize the definition of Δ from the root system Φ of type A_{n-1} to any finite (real) reflection group W.

The root system, positive roots, simple roots, and almost positive roots $\Phi_{\geq -1}$ still make sense as above. The *cluster complex* Δ is again the clique complex on vertex set $\Phi_{\geq -1}$, having as its 1-skeleton a graph defined by a certain compatibility relation on almost positive roots. Subsets of $\Phi_{\geq -1}$ whose elements are pairwise compatible are called *clusters*.

The compatibility relation is easily described when one of the roots is a negative simple root $-\alpha_i$: an almost positive root β is compatible with $-\alpha_i$ if and only if the unique expansion $\beta = \sum_i c_j \alpha_j$ into simple roots α_j has $c_i = 0$. One can then extend the definition of compatiblity to *all* pairs of almost simple roots using an action of a dihedral group on $\Phi_{\geq -1}$ (corresponding to the action of the symmetries of the polygon P above). Every almost positive root has at least one negative simple root in its orbit under this dihedral action.

The dihedral action comes from a *deformation* of the usual dihedral action on a 2-plane arising in the theory of the Coxeter element, using the two-coloring of the Coxeter generators S that comes from the Coxeter diagram being a tree. For a more precise description of this dihedral action, see [4, §4.3].

Case-by-case verifications have shown that the number of clusters has cardinality

$$\frac{1}{|W|} \prod_{i=1}^{r} (e_i + h + 1)$$

where e_i are the *exponents* of W, and h is the *Coxeter number*.

References

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