

# SIMPLICES AND SPECTRA OF GRAPHS

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ABSTRACT. In this note we show the  $n - 2$ -dimensional volumes of codimension 2 faces of an  $n$ -dimensional simplex are algebraically independent functions of the lengths of edges. In order to prove this we compute the complete spectrum of a combinatorially interesting graph.

## INTRODUCTION

Let  $T_n$  be the set of congruence classes of simplices in Euclidean space  $\mathbb{E}^n$ . The set  $T_n$  is an open manifold (also a semi-algebraic set) of dimension  $(n + 1)n/2$ . Coincidentally, a simplex  $T \in T_n$  is determined by the  $(n + 1)n/2$  lengths of its edges. Furthermore, the square of the volume of  $T \in T_n$  is a polynomial in the squares of the edgelengths. This polynomial is given by the Cayley-Menger determinant formula:

$$V^2(T) = \frac{(-1)^{n+1}}{2^n(j!)^n} \det C,$$

where  $C$  is the *Cayley-Menger matrix*, defined as follows:

$$C_{ij} = \begin{cases} 0, & i = j \\ 1, & \text{if } i = 1 \text{ or } j = 1, \text{ but not both} \\ l_{(i-1)(j-1)}^2 & \text{otherwise} \end{cases}$$

Note that an  $n$ -dimensional simplex also has  $(n + 1)n$   $n - 2$  dimensional faces, and so the following question is natural:

*Question 1.* Is the congruence class of the simplex  $T$  determined by the  $n - 2$ -dimensional volumes of the  $n - 2$ -dimensional faces?

Question 1 must be classical, but the first reference that I am aware of is Warren Smith's Princeton PhD thesis [5].

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In the AIM workshop on Rigidity and Polyhedral Combinatorics Bob Connelly (who was unaware of the reference [5]) raised the following:

*Question 2.* Is the *volume* of the simplex  $T$  determined by the  $n - 2$ -dimensional volumes of the  $n - 2$ -dimensional faces?

In fact, Connelly stated Question 2 for  $n = 4$ , which is the first case where the question is open (for  $n = 3$  the answer is trivially "Yes", since  $3 - 2 = 1$ , and we are simply asking if the volume of the simplex is determined by its edge-lengths. In dimension 2, the answer is trivially "No", since  $2 - 2 = 0$ , and the volume of codimension-2 faces of a triangle carries no information.

Clearly, the affirmative answer to Question 1 implies an affirmative answer to Question 2. At the time of this writing, both questions are open. In this note we show

**Theorem 3.** *The  $(n + 1)n/2$   $n - 2$ -dimensional volumes of the  $n - 2$ -dimensional faces of an  $n$ -dimensional simplex are algebraically independent over  $\mathbb{C}[l_{12}, \dots, l_{((n+1)(n+1)}]$ .*

Theorem 3 is clearly a necessary step in the direction of resolving Question 1, but is far from sufficient. To show it, consider the map of  $\mathbb{R}^{(n+1)n/2}$  to  $\mathbb{R}^{(n+1)n/2}$ , which sends the vector  $E$  of edge-lengths to the vector  $F$  of areas of  $n - 2$ -dimensional faces. To show Theorem 3 it is enough to check that the Jacobian  $J(E) = \partial F / \partial E$  is non-singular at *one* point. We will use the most obvious point  $p_1$  : one corresponding to a regular simplex with all edge-lengths equal to 1. By symmetry considerations, the Jacobian  $J(p_1)$  can be written as

$$J(p_1) = cM,$$

where

$$M_{ij} = \begin{cases} 1, & \text{if the } j\text{-th edge is incident to the } i\text{-th face of dimension } n - 2. \\ 0, & \text{otherwise} \end{cases}$$

The first observation is

**Lemma 4.** *The constant  $c$  above is not equal to 0.*

*Proof.* This follows from the observation that the volume of a  $k$ -dimensional simplex is a *homogeneous* function of the edge-lengths, of degree  $k$ . An application of Euler's Homogeneous Function Theorem shows that at  $p_1$ ,

$$\frac{\partial F_i}{\partial e_j} = \begin{cases} \frac{2}{n-1}F, & \text{if } e_j \text{ is incident to } F_i \\ 0, & \text{otherwise,} \end{cases}$$

where  $F$  is the common value of the  $n - 2$ -dimensional volume of the  $F_i$ , which implies that  $c = \frac{2}{n-1}F$ .  $\square$

Theorem 3 thus reduces to the assertion that the determinant of the matrix  $M$  is not zero. We will be able to prove a much stronger result (of interest in its own right):

**Theorem 5.** *The singular values of  $M$  are as follows: The value 1 appears with multiplicity  $(n + 1)(n - 1)/2$ . The value  $(n - 2)$  appears with multiplicity  $n$ . The value  $(n - 2)(n - 1)$  appears once.*

**Corollary 6.** *The absolute value of the determinant of  $M$  equals*

$$(n - 2)^{n+1}(n - 1) \neq 0,$$

for  $n > 2$ .

To prove Theorem 5, first recall that the singular values of  $M$  are the positive square roots of the eigenvalues of  $N = MM^t$ . In its turn,  $N$  has rows and columns indexed by  $n - 1$ -subsets of  $R_{n+1} = \{1, \dots, n + 1\}$ . The  $ij$ -th element of  $N$  equals the number of 2-element subsets the  $i$ -th and the  $j$ -th two element subsets  $s_i$  and  $s_j$  have in common. This, in turn, can be written as follows:

$$N_{ij} = \begin{cases} n(n - 1)/2, & i = j \\ (n - 1)(n - 2)/2, & |s_i \cap s_j| = n - 2 \\ (n - 2)(n - 3)/2, & |s_i \cap s_j| = n - 3. \end{cases}$$

The matrix  $N$  is the adjacency matrix of a multi-graph  $G_N$ , which has a rather large symmetry group. These symmetries will allow us to obtain the complete spectral decomposition of the matrix  $N$ . Indeed, the symmetry group of  $G_N$  is the symmetric group  $S_{n+1}$ , while the stabilizer  $\Gamma_i$  of a vertex  $s_i$  is the group  $S_{n-1} \times S_2$  (the first factor acts on  $s_i$  itself, the second on  $R_{n+1} \setminus s_i$ .) The action of  $\Gamma_i$  on  $G_N$  has three orbits. The first consists of  $s_i$  itself. The second consists of all  $s_j$  such that  $|s_i \cap s_j| = n - 2$ , the third of all  $s_j$  such that  $|s_i \cap s_j| = n - 3$ .

At this point it behooves us to recall the concept of *graph divisor*.

### 1. GRAPH DIVISORS

The concept of graph divisor is discussed at great length in the books [2] and [1]

**Definition 7.** *Given an  $s \times s$  matrix  $B = (b_{ij})$ , let the vertex set of a (multi)graph  $G$  be partitioned into (non-empty) subsets  $X_1, X_2, \dots, X_s$ , so that for any  $i, j = 1, 2, \dots, s$ , each vertex from  $X_i$  is adjacent to exactly  $b_{ij}$  vertices of  $X_j$ . The multidigraph  $H$  with adjacency matrix  $B$  is called a front divisor of  $G$ .*

The importance of this concept to our needs is that the characteristic polynomial of a graph divisor *divides* the characteristic polynomial of the adjacency matrix of  $G$ . (hence the name). The most interesting (to us, anyway) example of a graph divisor

arises by having a subgroup  $\Gamma$  of the automorphism group of  $G$ . The quotient of  $G$  by  $\Gamma$  is the divisor we consider. Every eigenvector of  $\Gamma \backslash G$  lifts to an eigenvector of  $G$  with the same eigenvalue.

If we consider our graph  $G_N$  and the action of  $\Gamma_i$ , we observe (after a rather tedious computation) that the front divisor corresponding to the action of  $\Gamma_i$  on  $G_N$  has adjacency matrix

$$D = \begin{pmatrix} \frac{1}{2}(n-2)(n-1) & (n-3)(n-2)(n-1) & \frac{1}{4}(n-4)(n-3)(n-2)(n-1) \\ \frac{1}{2}(n-3)(n-2) & n^3 - 7n^2 + 17n - 14 & \frac{1}{4}(n^4 - 10n^3 + 39n^2 - 70n + 48) \\ \frac{1}{2}(n-4)(n-3) & n^3 - 8n^2 + 23n - 24 & \frac{1}{4}(n^4 - 10n^3 + 43n^2 - 90n + 76) \end{pmatrix}$$

A simple computation shows that the eigenvalues of  $D$  are  $(n-1)^2(n-2)^2$ ,  $(n-2)^2$ ,  $1$ , while the corresponding eigenvectors are (respectively):

$$\begin{aligned} & (1, 1, 1), \\ & ((1-n)/2, (3-n)/4, 1), \\ & ((2-3n+n^2).2, (2-n)/2, 1). \end{aligned}$$

Graph divisors are a very useful tool, but they have too obvious shortcomings:

- (1) Not all eigenvalues of the graph  $G$  are captured by the graph divisor.
- (2) We have no information on the *multiplicity* of any of the eigenvalues that are captured.

Here, however, we have a *deus ex machina* in the form of

## 2. GELFAND PAIRS

We will not need any more than the (rather little) presented in Diaconis' little book [3, page 54]

First:

**Definition 8.** Let  $G$  be a group acting transitively on a finite set  $X$  with isotropy group  $N$ . A function  $f : G \rightarrow \mathbb{C}$  is called  $N$ -bi-invariant if  $f(n_1gn_2) = f(g)$ , for all  $n_1, n_2 \in N, g \in G$ . The pair  $G, N$  is called a Gelfand pair if the convolution of  $N$ -bi-invariant functions is commutative.

In our application,  $G = S_{n+1}$ ,  $N = S_{n-1} \times S_2$ , and  $X = G_N$ .

The further results we need are the following (the citations are from [3]):

**Lemma 9** (Lemma 5, page 53). Let  $\rho, V$  be an irreducible representation of the finite group  $G$ . Let  $N \subset G$  be a subgroup and let  $X = G/N$  be the associated homogeneous space. The number of times that  $\rho$  appears in  $L^2(X)$  equals the dimension of the space of  $N$ -fixed vectors in  $\rho, V$ .

**Theorem 10** (Theorem 9, page 54). *The following three conditions are equivalent:*

- (1)  $G, N$  is a Gelfand pair.
- (2) The decomposition of  $L^2(X)$  into irreducible representations of  $G$  is multiplicity-free.
- (3) For every irreducible representation  $(\rho, V)$  there is basis of  $V$  such that  $\hat{f}(\rho) = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$ , for all  $N$ -bi-invariant functions  $f$ .

The main significance (as seen by this author) of the Gelfand pair property is that the eigenvalues of  $N \backslash G / N$  (which is the "front divisor"  $D$ ) are the same as the eigenvalues of  $X = G / N$ , since each irreducible factor of  $L^2(X)$  contains an invariant function (as per Lemma 9)

We will finally need the following (attributed by Diaconis to [4]):

**Fact 11.** *The pair  $S_{n+1}, S_{n-1} \times S_2$  is a Gelfand pair. The dimensions of the irreducible representations of  $G_n$  are  $(n + 1)n/2, n, 1$ .*

We now have everything we need to finish the proof of Theorem 5, and hence Theorem 3. Since the graph laplacian (or adjacency matrix) commutes with the action of the automorphism group, each eigenspace of the adjacency matrix  $M$  is a sum of irreducible representations of  $S_{n+1}$ . By Fact 11 there are precisely three eigenspaces, and their dimensions are as stated in Theorem 5. The only question is how to decide which eigenspace has which dimension. Since we know the eigenvectors of the "front divisor" matrix  $D$  (see the end of the Introduction) we can match them up with the "spherical functions" of the Gelfand pair  $S_{n+1}, S_{n-1} \times S_2$  (as given on [3, page 57], whence the result follows.

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