1 Appendix

This appendix augments the paper [BFHRS]. Let

$$S_2 = \begin{bmatrix} 1 & \frac{1}{2}+\frac{\sqrt{5}}{2} & 0 \\ 1 & 1 & 1 \\ 1 & -\frac{1}{2}+\frac{\sqrt{5}}{2} & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & \frac{1}{2}+\frac{\sqrt{5}}{2} & 1 \\ 1 & 1 & \frac{3}{2}-\frac{\sqrt{5}}{2} \\ 1 & -\frac{1}{2}+\frac{\sqrt{5}}{2} & -\frac{1}{2}+\frac{\sqrt{5}}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Corollary 2.7 of [BFHRS] asserts that

$$\mathrm{mr}(\mathcal{M}^{\mathbb{Q}}_{C_{S_2}}) = 9 > 8 = \mathrm{mr}(\mathcal{M}^{\mathbb{R}}_{C_{S_2}}).$$

and thereby provides a counterexample to the central conjecture in [AHKLR, pp. 112-113],

> In this paper we raise the following basic conjecture. For any $m \times n$ sign pattern matrix A with $\mathrm{mr}(A) = k$, there exists a rational matrix (equivalently, an integer matrix) $B \in \mathcal{Q}(A)$ such that rank $B = k$.

With our notation, this would be:

> For any $m \times n$ sign pattern matrix Z with $\mathrm{mr}(\mathcal{M}^{\mathbb{R}}_Z) = k$, there exists a rational matrix (equivalently, an integer matrix) B in the sign pattern class of Z such that rank $B = k$.

The sign-pattern class restricts the signs of the entries, a stronger restriction than restricting the zero-nonzero pattern. Thus we have

Counterexample 1.1. Let A be a realization of $C^{\mathbb{R}}_{S_2}$ of rank 8, and let $Z_{C_{S_2}}$ be the sign pattern of A. By Corollary 2.7 of [BFHRS] there is no rational matrix with sign pattern Z of rank 7. Hence the minimum rank among the rational matrices with sign pattern Z is larger than the minimum rank among the real matrices with sign pattern $Z_{C_{S_2}}$.

We now construct an explicit realization of a sign pattern $Z_{C_{S_2}}$ (see below) having real minimum rank 8 that does not have a rational realization of rank 8 from S_2 using *Mathematica* as follows:

1. The matrix S_2 was entered (in exact arithmetic using (-1 + Sqrt[5])/2), as were the following utilities.

```
zero[m_, n_] := Table[Table[0, {j, 1, n}], {i, 1, m}];
submtx[A_, setr_, setc_] := Block[{M1 = A[[setr]]},
      Transpose[Transpose[M1][[setc]]]];
```

2. A 25×3 matrix C_3 of 3-cycles was created with the command

```
C3 = zero[25, 3];
c = 0;
Do[Do[
  Do[If[Det[submtx[S2, {i, j, k}, {1, 2, 3}]] == 0,
     c = c + 1; C3[[c, 1]] = i; C3[[c, 2]] = j; C3[[c, 3]] = k],
   {k, j + 1, 11}],
 {j, i + 1, 11}], {i, 1, 11}]
```

3. A 145×4 matrix C_4 of 4-cycles were created with the command

```
C4x = Subsets[{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}, {4}];
C4 = C4x; Do[s = C4x[[k]];
Do[If[Dimensions[Intersection[C3, {Delete[s, i]}]][[1]] == 1,
    C4 = Complement[C4, {s}]], {i, 1, 4}],
 {k, 1, Dimensions[C4x][[1]]}]
```

4. The following commands were executed to create a 170×11 real matrix $A \in \mathcal{M}^{\mathbb{R}}_{C_{S_2}}$ of rank 8 (forcing A to have the columns of S_2 in its nullspace).

```
A = zero[170, 11];
Do[Clear[x, y, z, u, sol];
   sol =
   Solve[S2[[C3[[k, 1]]]] + y*S2[[C3[[k, 2]]]] + x*S2[[C3[[k, 3]]]] ==
                                                    {0, 0, 0}, {x, y}];
   x = x /. sol[[1]]; y = y /. sol[[1]];
   A[[k, C3[[k, 1]]]] = 1; A[[k, C3[[k, 2]]]] = y; A[[k, C3[[k, 3]]]] = x,
 {k, 1, 25}];
Do[kk = k + 25; M = {S2[[C4[[k, 1]]]], S2[[C4[[k, 2]]]], S2[[C4[[k, 3]]]]};
    m = Det[M];
    M1 = {S2[[C4[[k, 4]]]], S2[[C4[[k, 2]]]], S2[[C4[[k, 3]]]]};
    m1 = Det[M1]/m;
    M2 = {S2[[C4[[k, 1]]]], S2[[C4[[k, 4]]]], S2[[C4[[k, 3]]]]};
    m2 = Det[M2]/m;
    M3 = {S2[[C4[[k, 1]]]], S2[[C4[[k, 2]]]], S2[[C4[[k, 4]]]]};
    m3 = Det[M3]/m;
    A[[kk, C4[[k, 1]]]] = m1;
    A[[kk, C4[[k, 2]]]] = m2;
    A[[kk, C4[[k, 3]]]] = m3;
    A[[kk, C4[[k, 4]]]] = -1,
  {k, 1, 145}]
```

5. The sign pattern $Z_{C_{S_2}}$ is the sign pattern of A (broken into two halves of 85 rows each) is presented on the following page:

References

[AHKLR] M. Arav, F. Hall, S. Koyucu, Z. Li and B. Rao. On Rational Realizations of the Minimums Rank of a Sign Pattern Matrix. *Linear Alg. Appls.*, 2005, 411:11–125.

[BFHRS] Avi Berman, Shmuel Friedland, Leslie Hogben, Uriel G. Rothblum, and Bryan L Shader. Minimum rank of matrices described by a graph or pattern over the rational, real and complex numbers, submitted to *Electronic Journal of Combinatorics*.

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