Workshop on FREE ANALYSIS
June 19, 2006 - June 23, 2006
Organizers: Dimitri Shlyakhtenko and Dan Voiculescu

The theory of von Neumann algebras is a prominent part of theoretical mathematics and also of mathematical physics. It is the modern language for formulating ideas of quantum mechanics, and it is also the source of important ideas for modern analysis. The group that gathered at AIM this week consisted of a diverse collection of mathematicians—with broadranging interests—who wanted to be brought up to speed on fastbreaking new ideas in this subject area.

Of particular interest are techniques, pioneered by organizer Voiculescu, of developing a highly noncommutative parallel to classical probability theory and applying it to important structural problems of von Neumann algebras. It turns out that this approach is particularly well suited for the von Neumann algebras of (noncommutative) free groups.

At the origin of these applications is the realization that the large N limit of an n-tuple of independent Gaussian random matrices (in the sense of increasing size N) provides an asymptotic model for the von Neumann algebra of a free group on n generators. At the same time, the von Neumann algebras of free groups are the most basic objects for the new noncommutative probability theory in which independence mimics free products instead of tensor products (free independence). In particular, random matrices in the large N limit constitute a bridge between classical probability theory and the new free probability theory.

In free probability, independent random variables, called ***free random variables***, are very far from commuting in general. The Wigner semicircle law makes its appearance in the free Central Limit Theorem as the replacement for the Gauss law. More generally, there are free convolution operators for probability distributions and a free analogue of the Levy-Khintchine theorem with the same parametrization of laws as in the classical case, but with a shrinking of supports of the free laws in this correspondence.

One more recent development in free probability is the emergence of a free entropy theory. This again has produced a new line of applications to von Neumann algebras and led to the solutions of some longstanding open problems. In another direction, the free entropy also has important connections to large deviations for random matrices thanks to the work of P. Biane, A. Guionnet and others.

A new development in von Neumann algebras is the emerging theory of L^2 invariants designed to extend L^2 Betti numbers from groups to operator algebras (work of A. Connes and D. Shlyakhtenko). Quite surprisingly, there exists a connections to a free entropy dimension quantity arising in free probability theory.

On the other hand, there is a combinatorial side to free probability. This was initiated by Roland Speicher, who found that noncommutative moment computations in free probability run parallel to classical ones if the lattice of all partitions of $\{1, 2, \ldots, n\}$ is replaced by the lattice of nonvanishing partitions.

The idea behind the title of the workshop—"Free Analysis"—came from the realization that many subjects connected with free probability have naturally required the development of a kind of non-commutative differential calculus. At its heart are the free difference quotient derivations. These are highly non-commutative generalizations of the the difference quotient

$$\partial f(s,t) = \frac{f(s) - f(t)}{s - t}.$$

It is exactly this calculus of derivations that appears in the differential operators that govern the asymptotics and large deviations of random matrices as well as the non-commutative stochastic differential equations that relate to free Brownian motion and free entropy; these same derivations occur in connection with L^2 Invariants. All of these subjects run into deep analytical problems related to this calculus of non-commutative differential operators, i.e., into *free analysis.*

Very roughly, the calculus of these derivations is a non-commutative analog of the classical differential calculus (where the derivations are the familiar partial derivatives $\partial f/\partial x_j$). Many questions from classical analysis (such as, for example, the existence of eigenvectors $f(x) = \exp(x)$ for the classical differentiation operator, crucial for Fourier analysis) have natural analogues in the non-commutative calculus (here the similar role is played by generalized resolvents whose properties mimic the fact that if $G(x) = (z - x)^{-1}$ then the difference quotient derivation ∂ satisfy a corepresentation property

$$\partial G(s,t) = G(s)G(t).$$

Hence the central subject of the workshop was to collect various analytical facts about the “free differential calculus” and explore the available properties/facts/conjectures/techniques that would be the subject of “free analysis”. Advances in the subject of free analysis are therefore of foremost importance in all of the fields mentioned above.

As is natural in analysis, one important subject is the study of inequalities; this was the starting point for the workshop. These include Poincaré and log-Sobolev-type inequalities which occur both in random matrix theory and free probability theory. Already on the first day of the workshop, the vigorous discussions resolved some open questions about such inequalities.

The theory of von Neumann algebras has been a fruitful source of ideas for many parts of mathematics, and proves to be a very useful framework for formulation of non-commutative analysis. Just a few years ago Alain Connes used it to create noncommutative geometry. The ideas treated in the present workshop illustrate the profound connections with classical analysis. As the workshop has shown, there will be many new developments for the foreseeable future.