Research Statement

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I work in the field of number theory. It is a common and successful practice in number theory to encode part of the information contained in an arithmetic object in form of an analytic L-function. The best known example of an L-function is the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\,prime} (1 - \frac{1}{p^s})^{-1} \qquad \text{for } \mathrm{Re}(s) > 1.$$

The Riemann zeta function provides rich number-theoretic information. For example, using the divergence of the sum at $s = 1$ and the product formula, Euler gave another proof of the fact that there are infinitely many prime numbers. The non-vanishing of $\zeta(s)$ (or its meromorphic continuation) at all points on the line $\mathrm{Re}(s) = 1$ is crucial in the proof of the Prime Number Theorem, which states that the number of primes less than or equal to x grows as $x/\log(x)$. The importance of L-functions is further highlighted by the fact that two of the seven millennium problems posed by the Clay Mathematics Institute are related to L-functions — the *Riemann hypothesis* concerning the zeros of the Riemann zeta function, and the *Birch and Swinnerton-Dyer conjecture* about the behavior of the L-function associated to an elliptic curve.

There are two main directions in the field closely related to my research. The first one is the *Langlands Program*, sweeping conjectures made by Robert Langlands about "automorphic representations" and their L-functions. The main idea is the following — given an L-function occuring naturally in a number theoretic context, construct an automorphic representation whose L-function is exactly the one you want to study. The motivation for this idea is that, on the automorphic side, there are available a wide array of techniques involving representation theory, algebraic geometry and many other fields of mathematics which should help get new information on the L-function. This strategy was successfully used by Andrew Wiles and others to show that the L-function associated to an elliptic curve defined over rational numbers, is the same as the L-function obtained from automorphic forms. This led to the proof of Fermat's Last Theorem and special cases of the Birch and Swinnerton-Dyer conjecture. Another important feature of the Langlands program is that if two groups are related to each other then Langlands conjectures a relation between their automorphic representations, thereby allowing transfer of information from one group to the other.

The second direction is *Deligne's conjecture* on special values of L-functions. Roughly speaking, in the context of automorphic forms, Deligne conjectured that the special value of an L-function associated to an automorphic representation at a critical point is algebraic up to a specified transcendental number. An instance of this conjecture is the well known fact that $\zeta(2k)/\pi^{2k}$ is a rational number for any positive integer k. This conjecture, on the one hand, contains rich information on the automorphic representation and, on the other hand, the algebraic part has applications to several fields such as p-adic L-functions and congruences of modular forms.

There is a lot of work towards the above two problems in the literature for automorphic representations that have a Whittaker model i.e. are *generic.*. For instance, in [CKP-SS], the authors have obtained the Langlands transfer of generic representations of classical groups to GL_n using the converse theorems of [CP-S2] and the Langlands-Shahidi method. In [R], special value results for the L-functions of generic $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ are derived using Rankin Selberg integrals.

A substantial portion of automorphic representations for classical groups are not generic and I am interested in understanding the methods and results for these representations. One such class of non-generic representations are the cuspidal, automorphic representations π_F of the symplectic group $\mathrm{GSp}_4(\mathbb{A})$ obtained from holomorphic Siegel modular forms F of degree 2. The Siegel modular forms have a long history, both classically and in a representation-theoretic sense. It also has the advantage that the representation π_F has a Bessel model and being a low rank case one can do explicit computations. The properties of these representations π_F are closely related to the family of L-functions $L(s, \pi_F \times \tau)$, where τ is any cuspidal, automorphic representation of GL_2. The two main problems regarding Siegel modular forms motivating my work are:

Langlands functorial transfer of π_F to GL_4: To prove this, one method is the converse theorem [CP-S1], which requires information on analytic properties of $L(s, \pi_F \times \tau)$, where τ is any cuspidal representation of GL_1 or GL_2 (see [AS] for the generic case).

Deligne's special value conjectures: In [De], Deligne conjectures the algebraicity of certain special values of the L-function $L(s, \pi_F \times \tau_f)$, where $f \in S_k(N, \chi)$.

My approach to study these L-functions is via integral representations. The idea is to consider an integral which can be written as a product of local integrals, the product taken over all primes. Since the L-functions share the same feature, one expects the integral to evaluate to an L-function. The motivation for obtaining an integral representation for an L-function is that one hopes to understand the analytic and arithmetic properties of the integrand well enough to transfer these properties to the L-function. Using the integral $Z(s)$ defined by Furusawa in [Fu], we prove the following.

Theorem 1. (Pitale [P4], Pitale-Schmidt [PiSc3], [PiSc5]) *Let F be a degree 2 cuspidal Siegel Hecke newform of weight l with respect to the Borel congruence subgroup of square-free level M. Let τ be any irreducible, cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. Then a choice of local vectors can be made so that the integral $Z(s)$ defined in [Fu] is equal to* $L(3s + \frac{1}{2}, \pi_F \times \tau)$.

Using the arithmetic properties of the Eisenstein series involved in the integral $Z(s)$ in Theorem 1, we get the following case of Deligne's conjecture.

Theorem 2. (Pitale [P4], Pitale-Schmidt [PiSc3], [PiSc5]) *Let F be as in Theorem 1. Let f be a holomorphic, cuspidal Hecke eigenform of weight l, any level N and nebentypus χ. Then $L(\frac{l}{2} - 1, \pi_F \times \tau_f)/\Big(\pi^{5l-8}(F,F)_2(f,f)_1\Big)$ is algebraic.*

In a joint work with Abhishek Saha (ETH Zurich) and Ralf Schmidt (University of Oklahoma), we obtain a pullback formula for $Z(s)$ to $\mathrm{GU}(3,3)$, which gives us the following analytic properties of the L-function.

Theorem 3. (Pitale-Saha-Schmidt [PiSaSc]) *Let F and f be as in Theorem 2, with $M = 1$. Then the poles of $L(s, \pi_F \times \tau_f)$ in the right half plane $Re(s) > 0$ are at most simple and are contained in the set $\{1, \frac{3}{2}, 2\}$.*

Current projects: The two main questions that I am currently working on are to generalize Theorem 1 to include arbitrary Siegel modular forms and to apply the theorem towards the Langlands functorial transfer and Deligne's conjecture. With regards to generalization, we observe that one of the key ingredients in the integral $Z(s)$ in Theorem 1 is explicit formulas for *test vectors* in a

Bessel model for the local representations of GSp_4. The test vectors are vectors for which the Bessel functional evaluates to a non-zero number. Information on these test vectors is known in very few cases — unramified representation ([Su]), Steinberg representation ([P4]) and holomorphic discrete series ([PiSc4]). In order to generalize the theorems to arbitrary $\mathrm{GSp}_4(\mathbb{A})$ representations, an important component is to obtain detailed information on the Bessel models for local representations. The work in [P4] is a first step towards that and I am currently working on this topic.

Concerning applications of Theorem 1, I am working on the following two problems. The first one involves *analytic properties* of the L-functions. This requires better understanding of the analytic properties of a degenerate $\mathrm{GU}(3,3)$ Eisenstein series. Another important step necessary for obtaining the Langlands transfer is the functional equation of the L-function, which would give the analytic properties of the L-function in the entire complex plane. I am currently working on these questions with Saha and Schmidt. The second question is regarding *special values* of the L-functions. I am working on obtaining Theorem 2 for all critical points, any weight and get detailed information on the algebraic part. This involves arithmetic properties of the Fourier coefficients of Eisenstein series and translating them, via the integral, to the L-function.

Classical modular forms: Another aspect of my research involves classical modular forms. Many of the applications of the theory of automorphic forms and representations to number theory are obtained by considering classical modular forms. It is the interaction between the classical methods and the underlying representation theory which turns out to be very fruitful and provides deep insights into the problems. I have worked on classical modular forms in three different settings — modular forms for the spin group, Siegel modular forms and Jacobi forms.

Modular forms for spin groups: In my Ph.D. thesis, I considered the problem of constructing classical modular forms on the group $\mathrm{GSpin}(1,4)$, which are counter-examples to the Ramanujan conjecture. Using the beautiful classical converse theorem due to Maass [Ma] and a representation-theoretic Waldspurger type formula, I obtain the following theorem.

Theorem 4. (Pitale [P1]) *Let f be a weight $1/2$ Maass form with respect to $\Gamma_0(4)$. There is an injective map $f \mapsto F$, where F is a modular form with respect to $\mathrm{GSpin}(1,4)(\mathbb{Z})$. This map is Hecke equivariant and the cuspidal automorphic representation π_F of $\mathrm{GSpin}(1,4)(\mathbb{A})$, corresponding to F, is a CAP representation, i.e., it violates the generalized Ramanujan conjecture.*

I would like to further investigate two questions regarding modular forms on the spin group — the analogue of the pullback formula for the Saito-Kurokawa lift as in [Ic] and Bocherer's conjecture for Fourier coefficients of Siegel cusp forms and their inner forms. These problems have been suggested to me by Professor Furusawa.

Siegel modular forms: Let F be a Siegel Hecke eigenform of degree n, weight $l \geq 2$ and level N. Let $\mu(m)$ be the Hecke eigenvalue of F for all integers m satisfying $(m,N) = 1$. Let $\pi_F = \otimes' \pi_p$ be the cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ corresponding to F. Using [CF], [An] and combining the techniques from representation theory of $\mathrm{GSp}_4(\mathbb{Q}_p)$ and the classical setting we obtain

Theorem 5. *Let notations be as above. Then we have*

Sign changes for Hecke eigenvalues: (Pitale-Schmidt [PiSc2]) *For $n = 2$, if F is not in the Maass space, then there exists an infinite set S_F of prime numbers such that if $p \in S_F$ then there* *are infinitely many r such that $\mu(p^r) > 0$ and infinitely many r such that $\mu(p^r) < 0$.*

Classical interpretation of Ramanujan conjecture: (Pitale [P3]) *For $n \geq 1$, p a prime coprime to N, we have that π_p is tempered if and only if, for any $\epsilon > 0$, there is a $C_\epsilon > 0$, such that $|\mu(p^r)| \leq C_\epsilon p^{r\left(\frac{nk}{2} - \frac{n(n+1)}{4} + \epsilon\right)}$, for all $r \geq 0$.*

Jacobi Maass forms: The theory of holomorphic Jacobi forms has been studied extensively in the last few decades. One of the important features of Jacobi forms is that they form a bridge between the space of elliptic modular forms and Siegel modular forms (see [EZ]). Analogous to Maass forms on GL_2, one would like to have a theory of non-holomorphic Jacobi forms. In [P2], I have developed such a theory. The motivation for this is the well known representation theory of the Jacobi group by Berndt and Schmidt in [BS]. I would like to further understand the relation between the Fourier Jacobi coefficients of the real analytic Siegel Eisenstein series of degree 2 and Jacobi Maass forms. Professor Bocherer and Professor Ibukiyama have suggested this problem to me.

References

[An] Andrianov, A.: *Euler products corresponding to Siegel modular forms of genus* 2. Russian Math. Surveys **29**, 45–116 (1974)

[AS] Asgari, M., Shahidi, F.: *Generic transfer from* GSp(4) *to* GL(4). Compos. Math. **142**, no. 3, 541–550 (2006)

[BS] Berndt, R., Schmidt, R.: *Elements of the Representation Theory of the Jacobi Group.* Progress in Mathematics **163**, Birkhauser Verlag Basel, (1998)

[CF] Chai, Ch.-L., Faltings, G.: *Degeneration of Abelian Varieties.* Springer, Ergebnisse d. Math. **22**, Berlin, Heidelberg New York, 1990

[CKP-SS] Cogdell, J., Kim, H., Piatetski-Shapiro, I., Shahidi, F. *Functoriality for the classical groups.* IHES **99**, 163-233 (2004)

[CP-S1] Cogdell, J., Piatetski-Shapiro, I.: *A Converse Theorem for* GL_4. Math. Res. Lett. **3**, 67–76 (1996)

[CP-S2] Cogdell, J., Piatetski-Shapiro, I.: *Converse theorems for GL_n,* II. J. Reine Angew. Math. **507**, 165-188 (1999)

[De] Deligne, P.: *Valeurs de fonctions L et periodes d'integrales.* Proc. Symp. Pure Math. **33**, part 2, 313–346 (1979)

[EZ] Eichler, M., Zagier, D.: *The theory of Jacobi forms.* Progress in Mathematics **vol. 55**, Birkhauser Boston Inc., (1985)

[Fu] Furusawa, M.: *On L-functions for* $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ *and their special values.* J. Reine Angew. Math. **438**, 187–218 (1993)

[Ic] Ichino, A.:*Pullbacks of Saito-Kurokawa lifts.* Invent. Math. **162**, 551-647 (2005)

[Ma] Maass, H.: *Automorphe Funktionen von meheren Veranderlichen und Dirichletsche Reihen.* Abh. Math. Sem. Univ. Hamburg **16**, no. 3-4, 72–100 (1949)

[P1] Pitale, A.: *Lifting from* $\widetilde{\mathrm{SL}}(2)$ *to* $\mathrm{GSpin}(1,4)$. Int. Math. Res. Not. **63**, 3919–3966 (2005)

[P2] Pitale, A.: *Jacobi Maass forms.* Abh. Math. Sem. Univ. Hamburg **79**, 87-111 (2009)

[P3] Pitale, A.: *Classical interpretation of the Ramanujan conjecture for Siegel cusp forms of genus n.* Manuscr. Math. **130**, Issue 2, 225-231 (2009)

[P4] Pitale, A.: *Steinberg representation of* GSp_4*: Bessel models and integral representation of L-functions.* submitted, 39 pages (2009)

[PiSc1] Pitale, A., Schmidt, R.: *Ramanujan type results for Siegel cusp forms of degree* 2. J. Ramanujan Math. Soc. Volume 22 No. 1, 87-111 (2009)

[PiSc2] Pitale, A., Schmidt, R.: *Sign changes of Hecke eigenvalues of Siegel cusp forms of degree* 2. Proc. of Amer. Math. Soc. **136**, 3831-3838 (2008)

[PiSc3] Pitale, A., Schmidt, R.: *Integral representation for L-functions for* $\mathrm{GSp}_4 \times \mathrm{GL}_2$. J. Number Theory **129**, 1272-1324 (2009)

[PiSc4] Pitale, A., Schmidt, R.: *Bessel models for lowest weight representations of* $\mathrm{GSp}_4(\mathbb{R})$. Int. Math. Res. Not. Vol 2009, No. 7, 1159-1212 (2009)

[PiSc5] Pitale, A., Schmidt, R.: *Integral representation for L-functions for* $\mathrm{GSp}_4 \times \mathrm{GL}_2$ II. submitted, 39 pages (2009)

[PiSaSc] Pitale, A., Saha, A., Schmidt, R.: *Pullback of Eisenstein series from* $\mathrm{GU}(3,3)$, in preparation (2009)

[R] Raghuram, A.: *On the special values of certain Rankin-Selberg L-functions and applications to odd symmetric power L-functions of modular forms.* Int. Math. Res. Not. (2009)

[Sa] Saha, A.: *L-functions for holomorphic forms on* $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ *and their special values.* Int. Math. Res. Not. Vol. 2009 No. 10, 1773–1837, 2009

[Su] Sugano, T.: *On holomorphic cusp forms on quaternion unitary groups of degree* 2. J. Fac. Sci. Univ. Tokyo **31**, 521–568 (1984)