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Constructions of Open Book Decompositions

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Constructions of Open Book Decompositions

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Dedicated to my brother, Isaac and my amazingly patient wife, Lorna. I am who I am because of you. Thanks for keeping me sane.

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Constructions of Open Book Decompositions

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We introduce the naive notion of a relative open book decomposition for contact 3-manifolds with torus boundary. We then use this to construct nice, minimal genus open book decompositions compatible with all of the universally tight contact structures (as well as a few others) on torus-bundles over S^1, following Honda's classification. In an accurate sense, we find Stein fillings of 'half' of the torus bundles. In addition, these give the first examples of open books compatible with contact structures without Stein fillings. We construct open books compatible with the universally tight contact structures on circle bundles over higher genus surfaces, as well, following a pattern introduced by a branched covering of B^4. Some interesting examples of open books without positive monodromy are emphasized.

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Chapter 1

Introduction

In 2000, Emmanuel Giroux began discussing a powerful relationship between the contact geometry of a 3-manifold and the topology of its fibered links. An idea first hinted at by a construction of Thurston and Winkelnkemper [32], showing that every fibration on a fibered link (that is, every open book decomposition) carries a contact structure. That the converse was also true was show in work of Torisu, through the study of contact Heegaard splittings. Such relationships between geometry and fibrations are paralleled in 4-dimensional symplectic manifolds in the work of Donaldson [2] and Gompf [18] interrelating symplectic structures and Lefschetz fibrations or pencils. In fact, for important examples symplectic manifolds X^4 with suitable compatible Lefschetz fibrations induce contact structures and compatible open book decompositions on their boundary 3-manifolds. Unlike the picture in 4-dimensions, Giroux's uniting of these ideas also included a description of how *all* open books compatible with a given contact structure must be related.

It is this completeness that makes the Giroux correspondence so powerful. It has led to the characterization of monodromies for tight contact structures as *Right-Veering* by Honda, Kazez and Matić [23] and to the Ozsváth-Szabó contact invariant in Heegaard Floer Homology [30], used to great fluency by Ghiggini, Lisca and Stipsicz [12], finishing Wu's classification [36] of tight contact structures on small Seifert-fibered spaces. More generally, it has been used to prove Property P and the surgery characterizations of the trefoil and figure 8 knots.

However, with all of the great theoretical implications, a relative few good examples are known. Using a trick of Akbulut and Ozbagci [1], one can turn any Legendrian surgery diagram into an open book decomposition. These often end up quite complicated and difficult to utilize. One measure of such complexity is in the *minimal genus* [7]. Knowing that your contact structure is compatible with an open book whose pages have genus 0 (so called, planar open books) give very strong constraints on the types of symplectic fillings it might admit. Etnyre [5] showed any filling must be negative definite. Further, the HF-contact invariant must be reducible [29]. It is still unclear (though potentially very useful) what higher genus might tell us. Minimal genus examples have been constructed by Schönenberger [31] for Lens spaces, using a technique known as *rolling up a diagram*. This technique has proved useful in constructing examples on graph manifolds as well [8]. However, until this paper, all known examples of open book decompositions compatible with tight contact structures use Stein-fillability to show tightness.

We construct genus 1 open book decompositions compatible with every universally tight contact structure on torus- and circle-bundles. Most of these have positive Giroux torsion and so do not admit strong (and hence Stein) fillings, though all of the universally tight contact torus bundles are at least weakly fillable. The open books also give us good information about the fillability of these contact manifolds. We use our examples to prove that roughly 'half' of these manifolds admit Stein fillings. Further, knowing these examples are genus 1 allows us to give suggestive evidence that some of these contact structures do not admit Stein fillings–we find open book decompositions whose monodromy cannot have a positive factorization. We provide examples of open books that are weakly but not strongly fillable as well as strongly but not Stein fillable (c.f. Section 4.7).

To do this, we introduce the relatively weak notion of a relative open book: an open book decomposition on a manifold with torus boundary. We prove the necessary compatibility and gluing theorems, and then construct an open book decomposition compatible with Honda's *basic slice*. From this, the rest of the examples for closed torus bundles are relatively straightforward, especially after a particularly nice relationship to the braid group on two letters is established. This parallels the behavior of basic slices nicely and adds a solid computational handle that was not as obvious before.

We finish in chapter 6 with by using branched coverings of B^4 to construct Stein-fillable open book decompositions of $S^1 \times \Sigma$ for higher genus surfaces. Once we have a few examples, it becomes straightforward to build open book decompositions compatible with all of the universally tight contact structures on circle bundles over Σ. We conjecture that these are also minimal genus, although the author hasn't an inkling how one might go about proving such a statement.

Chapter 2

Background

2.1 Preliminaries

We will assume a basic knowledge of the theory of contact structures (for a good introduction see [6] [28]). All manifolds will be oriented, all contact structures positive and coorientable. Unless otherwise noted, all manifolds will also be compact. For background on 4-manifolds and symplectic topology see [18], [27], [28].

2.2 Open Book Decompositions

We say an oriented link is nicely fibered if there is *any* fibration of the compliment where the closure of a fiber is a Seifert surface for the link. While this implies the existence of an open book decomposition, we use the latter term when referring to a particular fibration, rather than just the link itself. There are two ways of describing an open book: *embedded* and *abstract*.

Definition 2.2.1 (Embedded Version)**.** By an *open book decomposition* $\mathfrak{ob}$ of a 3-manifold M, we mean an oriented link $L \subset M$ and a fibration $\pi : M \backslash L \to S^1$. We require that this fibration be 'nice' in that the closure of every fiber, $Cl(\pi^{-1}(p))$, $p \in S^1$ is a Seifert surface for L. The link L is called the *binding* of the open book and the closure of each fiber $Cl(\pi^{-1}(p))$ is called a *page*.

Definition 2.2.2 (Abstract Version)**.** An *open book decomposition* is a pair (Σ, ϕ) where Σ is a bordered surface and the monodromy $\phi \in Aut^+$ is an orientation preserving automorphism of Σ. We require that ϕ restricts to the identity on a neighborhood of the boundary $\partial\Sigma$.

Note that any embedded open book also gives an abstract open book decomposition. The converse is not entirely true, as an abstract open book only determines an embedded open book *up to isomorphism*. We may choose a particular embedding by first forming the mapping torus $\Sigma \times I / ((x,1) = (\phi(x), 0))$ given by ϕ. We may then form the closed manifold $M = \Sigma \times_\phi S^1 \cup nS^1 \times D^2$, filling in each boundary component by gluing in a solid torus. Since ϕ restricts to the identity near the $\partial\Sigma$, there is a natural decomposition of each component of $\partial\Sigma \times_\phi S^1$, into $S^1 \times S^1$ given by the fiber and vertical S^1 directions. That is, under ϕ, every point $p \in \partial\Sigma$ traces out a factor S^1 in a T^2 component of $\partial\Sigma \times_\phi S^1$. We glue so that this circle bounds a disk in $S^1 \times D^2$. The other factor is given by a component of $\partial\Sigma$ and is glued to a longitude in $S^1 \times D^2$. These longitudes can be traced into the core of the solid torus, extending the fibration to the fibered link given by the cores of the solid tori. This *local neighborhood of the binding* is shown in Figure 2.1. The intuitive image to remember is a Rolodex, with the axis forming the binding of the open book and the pages radiating outward just as the cards do.

We can arrive at this picture in a more concise way. Again, form the mapping torus given by ϕ, but now collapse out each vertical circle $p \times S^1$ in each of

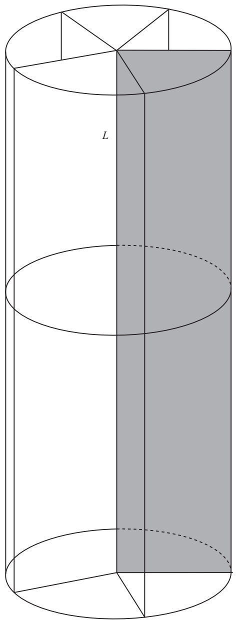


Figure 2.1: $S^1 \times D^2$ neighborhood of the binding in an open book. The top and bottom are identified.

the boundary tori. This forms a smooth manifold, each boundary torus collapses to a circle, the union of all forming the binding L.

Examples. S^3 has three very important open book decompositions with bindings the unknot and the positive and negative Hopf links $H^{\pm}$. The fibration of the unknot O is fairly easy to see: Take S^3 to be $\mathbb{R}^3 \cup \infty$ with the unknot lying in the y, z-plane. One fiber is the obvious disk bounded by O, also lying in the y, z-plane. To see the rest of the fibration, we think of blowing bubbles. Follow the disk as it blows out into larger and larger bubbles. It 'pops' to get the disk containing ∞ in the y, z-plane (the compliment of our first disk). The fibration then wraps around with larger bubbles shrinking back in. We can see this same fibration a different way. There is a Heegaard splitting of S^3 into two solid tori, S_1 and S_2, where S_2 is a tubular neighborhood of the unknot O. On S_1 we have the fibration by disks while on S_2 the fibration is given by the standard neighborhood in Figure 2.1 with boundary a longitude. Thus S^1 describes a mapping torus and S_2 is the solid torus filling in the binding.

The Hopf links have a slightly more interesting fibration which can also be visualized. To begin, we describe S^3 by starting with a tetrahedron and identifying sides. Let e and e' by two edges that do not share a vertex. Then at each of e and e', a pair of edges meet. We identify these edges, think of fanning out the tetrahedron to glue the left and right sides to get a thickened disk, and then wrapping the top around to meet the bottom. This indeed gives S^3. (One can see the genus one Heegaard splitting by slicing along a square separating e and e'. The square glues up into a torus and each component of the compliment glues up to be (the interior

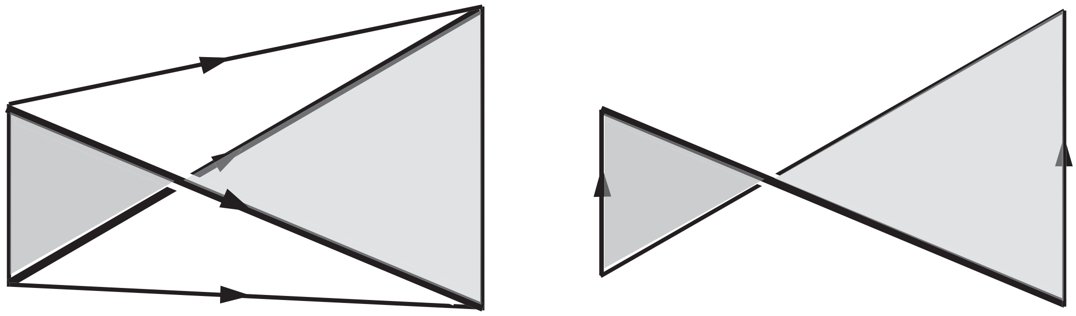


Figure 2.2: Fibration inside S^3 with binding the positive Hopf link H^+. The top and bottom faces, and front and back faces are identified in constructing S^3. Adjacent, a single fiber is shown, a half-twisted band with oriented boundary.

of) a solid torus.) The two fibrations are constructed by taking a band connecting e to e' with edges on opposite sides of the tetrahedron, as in Figure 2.2. All 4 edges are identified when forming S^3 and the band glues up to become an (twisted) annulus. Depending on which opposing pair of edges are chosen when building the band, the edges e and e' glue up to form either H^+ or H^-. The abstract open books for both H^+ and H^- are shown in Figure 2.3.

2.3 Murasugi Sum and Hopf Stabilization

While Murasugi sum is an operation on embedded fibered links, for ease we give only the abstract definition. Given two open books (Σ_i, ϕ_i) on 3-manifolds M_i and properly embedded arcs $\gamma_i \subset \Sigma_i$, for $i = 1, 2$ we form the Murasugi sum $(\Sigma, \phi) = (\Sigma_1, \phi_1)\natural(\Sigma_2, \phi_2) = (\Sigma_1\natural\Sigma_2, \iota_{1*}(\phi_2) \circ \iota_{2*}(\phi_2))$ as follows. Each arc γ_i has an rectangular neighborhood R_i with $\partial R = e_i, e_i', l_i, l_2'$ where the edges e_i and e_i' are on $\partial\Sigma_i$ and the arcs l_i and l_i' are parallel to γ_i. To form the new page $\Sigma_1\natural\Sigma_2$ we plumb Σ_1 to Σ_2 along the arcs γ_i, identifying R_1 and R_2 via an (orientation

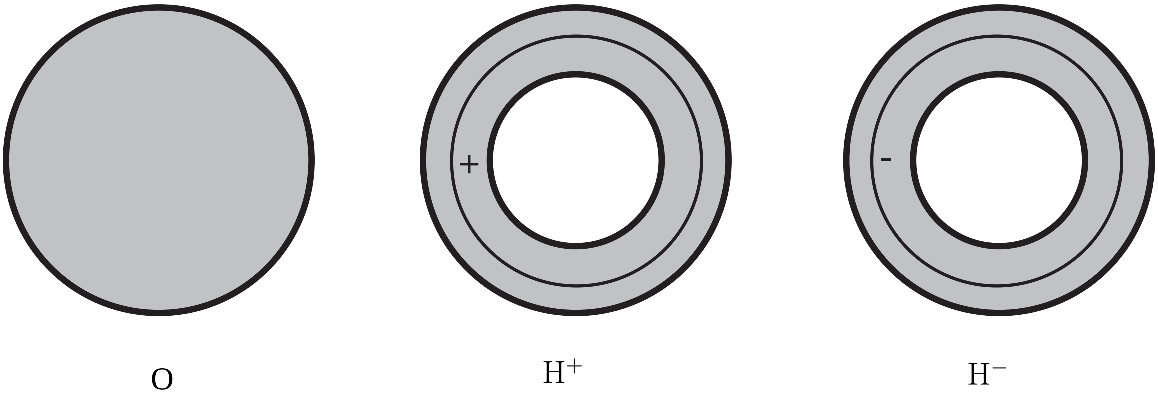


Figure 2.3: Abstract descriptions of the open books for O and $H^{\pm}$. The pages are the disk and annulus (resp.) with monodromies given by Dehn twists about the indicated curves.

preserving) map sending the edges e_1 and e_1' to the arcs l_2 and l_2' and similarly l_1, l_1' to e_2, e_2'. There are inclusion maps $\iota_i : \Sigma_i \to \Sigma_1 \natural \Sigma_2$ and we may construct an automorphism of the new page by having each ϕ_i act only on $\iota_i(\Sigma_i)$, with ϕ_1 acting first and then ϕ_2. This operation is symmetric but does depend on the choice of arcs γ_i. The ambient manifold M for the Murasugi sum (Σ, ϕ) is independent of these choices, however, and is homeomorphic to the connected sum $M_1 \# M_2$.

When (Σ_2, ϕ_2) is the open book given by the positive (resp. negative) Hopf link in S^3, we refer to the Murasugi sum operation as positive (resp. negative) *Hopf stabilization* or positive (resp. negative) *Hopf plumbing* . Since $M_2 = S^3$, this doesn't change the ambient manifold. In this case, there is a more concrete description of the Murasugi sum. Σ is given by adding a 1-handle to Σ_1 and $\phi = \phi_1 \circ D_\gamma$ where D_γ is a single positive (resp. negative) Dehn twist about any curve on Σ dual to the 1-handle (that is, γ runs over the 1-handle exactly once). This process can be reversed and is called *destabilization*. If (Σ, ϕ) is an open book where the monodromy can be written $\phi = \phi_1 \circ D_\gamma$, one can simultaneously remove the 1-handle and the Dehn twist D_γ and again arrive at a new fibered knot or link on the same manifold. In the interest of motivation we have the following theorem, proved by Giroux and Goodman [16] using the theory of contact structures.

Theorem 2.3.1 (Harer's Conjecture)**.** *Any fibered knot in S^3 can be obtained from the unknot by a sequence of positive and negative Hopf stabilizations.*

2.4 Compatibility and Giroux's Theorem

For some time now, connections between the world of open book decompositions have been found. A first indication was given by a construction by Thurston and Winkelnkemper showing how to assign a contact structure to an open book decomposition. Using later terminology, they show that any open book *carries* a contact structure. We say that a contact structure ξ and an open book decomposition $\mathfrak{ob}$ on a 3-manifold are *compatible* (or the open book *carries* the contact structure) if there exists a contact form α for ξ that restricts to a primitive form on each page of $\mathfrak{ob}$. Later work of Giroux [15] would show that this is true for any contact structure. This interplay of contact geometry and topology closely mirrors a similar pairing in 4-dimensions between symplectic structures and Lefschetz fibrations and pencils (c.f. Section 5).

Theorem 2.4.1 (Thurston-Winkelnkemper [32])**.** *Every open book decomposition $\mathfrak{ob}$ of a 3-manifold M carries a contact structure. Further, any two contact structures compatible with $\mathfrak{ob}$ are contact isotopic.*

Proof. We prove this in two stages, beginning with the easier 'existence' construction. Since compatibility is invariant under isomorphism (of both contact structure and open book) we use the abstract definition and build a contact structure in parts, first constructing a compatible contact structure on the mapping torus of the open book. One can then show how to patch in a standard solid torus $S^1 \times D^2$, though we will postpone this step as it is a corollary of a later lemma.

To that end, let (Σ, ϕ) be the abstract fiber and monodromy of the open book $\mathfrak{ob}$ on M. Let $\bar{M}$ be the mapping torus of ϕ and $\pi : \bar{M} \to S^1$ be the bundle projection. Let t be the coordinate on S^1 and, by abuse of notation, let $dt = \pi^*(dt)$. (This will form the vertical component of the contact form.) For the horizontal component, choose coordinates (r, θ) on each $[-1, 0] \times S^1$ end of Σ where $\partial\Sigma = \{0\} \times S^1$. Consider the set of primitive 1-forms on Σ which, are given by $(1+r)d\theta$ in these coordinates. This set is non-empty and convex. (For example, let Ω be a volume form on Σ given by $dr \wedge d\theta$ on the ends and total area $|\partial\Sigma|$. Let α be any 1-form modeled by $(1+r)d\theta$ on the ends. Then the 2-form $\Omega - d\alpha$ is 0 on the ends of Σ and is additionally exact. Write $\Omega - d\alpha = d\alpha'$ where α' is a 1-form which is 0 on the ends of Σ. Then $\beta = \alpha + \alpha'$ satisfies our criterion.) For any such β, $\phi^*(\beta)$ also satisfies these conditions and so we can form the 1-form on $\bar{M}$

$$\alpha_K = \sigma(t)\beta + (1 - \sigma(t))\phi^*(\beta) + K dt,$$

where σ is some smooth, increasing step function $\sigma : [0, 1] \to [0, 1]$, flat at $t = 0$ and $t = 1$.

If K is large enough, α_K is a contact form. Further, α_K restricts to a primitive 1-form on each fiber and so determines a contact structure which is compatible with the fibration on the mapping torus. It is possible (though we will postpone the details for now) to extend the contact structure in a way which is compatible with the open book decomposition. For now, we will say only that there is a suitable standard neighborhood of a transverse curve that is used to extend the contact structure across the $S^1 \times D^2$s.

To show that any two compatible contact structures are contact isotopic we need the following lemma.

Lemma 2.4.2. *Let $\pi : M \to S^1$ be a fiber bundle and α a contact form that restricts to a primitive 1-form on each fiber of π. Let dt be the pullback of any line form on S^1. Then for any $K > 0$ the form $\alpha_K = \alpha + Kdt$ is a contact form on M.*

Proof. The proof is an explicit calculation, noting that dt is closed and that $dt \wedge d\alpha$ is a volume form for M ($\ker(d\alpha)$ and $\partial/\partial t$ both point out positively transverse to the fibers). We have $d\alpha_K = d\alpha$ and so $\alpha_K \wedge d\alpha_K = \alpha \wedge d\alpha + Kdt \wedge d\alpha > 0$. $\square$

We need a slightly more careful assessment for the uniqueness result on an open book.

Lemma 2.4.3. *Let $\mathfrak{ob}$ be an open book decomposition of a 3-manifold M and α a compatible contact form. Choose coordinates $(\theta, r, \rho) \in S^1 \times D^2$ on (each component of) a neighborhood $\nu(L)$ of the binding and assume α restricts to 1-form $fd\theta$ on each page $\rho = c$ of the open book. There there is a smooth 1-form τ on M satisfying*

1. *On the compliment of a tubular neighborhood of the binding, τ can be given as the pullback of a volume form on S^1 under π.*

2. *On $\nu(L)$, $\tau = g(r)d\rho$ where $g(r)$ is a smooth step function modeled by r^2 near $r = 0$ and by $r = 1$ near $\partial\nu(L)$ and with $g'(r) > 0$.*

For any $K > 0$, $\alpha + K\tau$ is a contact form that restricts to a primitive form on the interior of every page of $\mathfrak{ob}$.

To prove the 'uniqueness' portion of Theorem 2.4.1, let α_0 and α_1 be two contact forms compatible with $\mathfrak{ob}$. We construct a homotopy of contact forms α_s between them and apply Gray's theorem. The associated contact structures are thus contact isotopic. The homotopy is constructed in three segments (and is through forms compatible with $\mathfrak{ob}$). We may need to first normalize each α_i near L to be of the form given in Lemma 2.4.3.

1. From $s = 0$ to $s = 1/3$ we take the straight line homotopy from α_0 to $\alpha_0 + K\tau$. By Lemma 2.4.3 this is through contact forms.

2. From $s = 1/3$ to $s = 2/3$ we interpolate between $\alpha_0 + K\tau$ and $\alpha_1 + K\tau$ (which can also be taken to be the straight line homotopy). For $K >> 0$ this is also through contact forms.

3. From $s = 2/3$ to $s = 1$ we again take the straight line homotopy, now from $\alpha_1 + K\tau$ to α_1.

□

Further relations were hinted at by Torisu [33], also noting uniqueness of a contact structure compatible with an open book but also noting that the operations of Murasugi sum for open books and connected sum for contact structures were compatible. Giroux finalized these connections with the following theorem (for a detailed proof see Goodman's thesis [20] or Etnyre's notes [6]). In particular, he specified exactly how *every* open book compatible with a given ξ must be related.

Theorem 2.4.4. *Giroux [15] For any contact structure ξ on a 3-manifold M, there is a compatible open book decomposition. If open book decompositions $\mathfrak{ob}_1$ and $\mathfrak{ob}_2$ carry isotopic contact structures ξ_1 and ξ_2, then there is a sequence of positive Hopf stabilizations $\mathfrak{ob}_i \rightsquigarrow \mathfrak{ob}_i'$ for $i = 1, 2$ such that $\mathfrak{ob}_1'$ and $\mathfrak{ob}_2'$ are isotopic.*

This is incredible and not terribly expected. According to Giroux, then,*all* of contact geometry on a 3-manifold M is determined by the topology of the fibered links on M. This has been an incredibly useful bijection and many powerful results have followed it. Honda-Kazez-Matić have shown that an open book decomposition is compatible with a a tight contact structure if and only if every open book decomposition is *Right-Veering*, which generalizes Goodman's sobering-arc technique [19]. The Ozsváth-Szabó contact invariant in Heegaard Floer Homology is constructed using open book decompositions. as well.

Given the powerful abstract results generated by this relationship, though, nice examples of open book decompositions are relatively scarce. Examples are known for Lens spaces [31] and Seifert-fibered spaces obtained by plumbing together disk bundles [4] [8], (though more continue to be found). We remark that, other than the examples produced here, all known open books compatible with tight contact structures have positive monodromies and so give Stein-fillable contact structures. Akbulut and Ozbagci [1] show how to turn any Legendrian surgery diagram into an open book decomposition, but this often results in highly complex and difficult to use diagrams.

In order to construct our examples, we introduce a naive but useful generalization of an open book decomposition, one that will allow us to construct compatible open book/contact pairs on manifolds with torus boundary. We are able to glue such objects and hence can find open book decompositions compatible with (in a reasonable sense) almost all tight contact structures on Torus bundles over S^1. (Which is to say, there are only finitely many that we cannot describe.)

Chapter 3

Relative Open Book Decompositions

Definition 3.0.1 (Embedded version)**.** Let M be a compact 3-manifold with boundary a disjoint union of tori. By a **relative open book decomposition** $\mathfrak{ob}$ of M, we mean an oriented link $L \subset M$ and a bundle $\pi : M \backslash L \to S^1$. We require this bundle be *nice* in that there exists a decomposition of each boundary torus $S^1 \times S^1$ in which π is a projection onto one factor. We also require a neighborhood $(0, 1] \times T^2 = S^1 \times \{D^2 \backslash 0\}$ of each component of L on which π is given by projection to ∂D^2. We call the closures of the fibers the *pages* of the open book and L the *binding*.

Definition 3.0.2 (Abstract version)**.** By a **relative open book decomposition** of a manifold M with torus boundary, we mean a pair (Σ, ϕ) consisting of a compact, bounded surface Σ and an orientation preserving automorphism $\phi \in \mathrm{Aut}(\Sigma)$. We require $\phi|_{\partial\Sigma} = Id$. We further require a partition of the components of $\partial\Sigma$ into *binding circles* and *boundary circles* and that they both be non-empty.

Unlike the closed case, here the abstract definition comes equipped with more information than the embedded. In particular, it gives a preferred decomposition of the boundary tori into page a vertical directions, giving natural identifications when gluing. To form the ambient manifold M, we proceed as we would for an open book on a closed manifold. First form the surface bundle $\Sigma \times_\phi S^1$. For each *binding* circle γ, we collapse the vertical slopes in the torus boundary traced by γ. The union of the image circles of these tori form the *binding* of the open book. The boundary circles are left alone to trace out boundary tori of M. We do this so the gluing of abstract open books makes sense without any additional information, that is without specifying which normal directions to the page match up. Since this has no affect on the associated contact structure, only the topology of the gluing, we leave this information out of the embedded definition. This makes the discussions in Chapter 4 much less cluttered and gives a better analogue of the decomposition into basic slices.

Definition 3.0.3. Let $\mathfrak{ob}$ be a relative open book decomposition of a manifold M with torus boundary and let ξ be a contact structure on M. We will say ξ is *compatible* with $\mathfrak{ob}$ if there exists a contact form α that restricts to a primitive 1-form on each page of $M \backslash L$. In addition, we have a compatibility condition on each boundary torus T of M. We require the foliation of T given by the contact structure be the same as that of the pages.

Proposition 3.0.5. *The space of contact structures compatible with a given relative open book with non-empty binding is non-empty and connected.*

Proof. This is exactly the Thurston-Winkelnkemper construction with the addition of a boundary condition. We still begin with a primitive 1-form β on Σ, but now with two models near $\partial\Sigma$ depending on whether one is looking at a boundary circle or a binding circle. On a neighborhood $[-1, 0] \times S^1$ of a binding circle, the model is the same and we require β to be given by $(1 + r)d\theta$. On a neighborhood of a binding circle, however, β should be modeled by $rd\theta$. The space of of such forms is non-empty and convex (as long as the set of binding circles is non-empty), and so we follow the procedures used in the closed case (Theorem 2.4.1) and they apply equally well here. In particular, the form τ constructed in Lemma 2.4.3 preserves the boundary condition when added to a compatible form α on a relative open book decomposition and so the homotopy induced by Gray's theorem preserves the foliation by pages. $\square$

For simplicity, we first begin with a gluing lemma that will prove useful in our construction of a compatible form.

Proposition 3.0.6 (Gluing, local)**.** *Let $\mathfrak{ob}_1$ and $\mathfrak{ob}_2$ be open book decompositions (or fiber bundles, if the set of binding circles is empty) on the 3-manifolds M_1 and M_2 (resp.). Let α_1 (resp. α_2) be a contact 1-form on M_1 (resp. M_2) that restricts to a primitive 1-form on each fiber of $\mathfrak{ob}_1$ (resp. $\mathfrak{ob}_2$). Let $T_1 \subset M_1$ and $T_2 \subset M_2$ be boundary tori. Suppose there exists smoothly varying coordinates (θ, r) on the $[-1, 0] \times S^1$ end of each fiber (associated to T_i) under which $\alpha_i = rd\theta$. Then there exists a smooth fibration $\mathfrak{ob}$ and 1-form α on $M = M_1 \cup_{T_1=T_2} M_2$ so that*

1. *α is a contact 1-form that restricts to a primitive 1-form on each fiber of $\mathfrak{ob}$*
2. *$\mathfrak{ob}$ restricts to $\mathfrak{ob}_i$ on each M_i.*
3. *$\ker \alpha = \ker \alpha_i$ on each M_i.*
4. *$\alpha = \alpha_i$ outside a neighborhood of $T = T_1 = T_2 \subset M$.*

Proof. The idea here is to interpolate between the two contact forms on each piece to construct a contact form on the whole. To this end, we must first extend the contact structures along T_i so we have some overlap to interpolate along. We may do this in such a way that α_i remains contact and restricts to $rd\theta$ on the extended $S^1 \times [-1, \epsilon)$ ends of the fibers. This gives us coordinates (z, θ, r) on the $S^1 \times S^1 \times I$ ends of the 3-manifolds M_i and we may glue by the map $M_1 \to M_2$ (where it makes sense) given by $(z, \theta, r) \to (z, -\theta, -r)$ (for $-\epsilon < r < \epsilon$. The restrictions of the forms α_1 and α_2 to the fibers agree under this identification and so, choosing any smooth increasing step function $\phi : (-\epsilon, \epsilon) \to [0, 1]$, we may make the form $\alpha = \phi(r)\alpha_1 + (1 - \phi(r))\alpha_2$, which restricts to a primitive form on the pages. Notice, we may also use this to fill in the standard neighborhoods in the Thurston-Winkelnkemper construction. As long as the fibration has a contact form modeled by $rd\theta$ near the ends $S^1 \times [t_0, t_1]$ and the prelagrangian tori we glue along have the same slope, the proposition still holds and we may still glue. We will see an example of this when constructing the open books on $S^1 \times \Sigma$ in Section 6.

□

Proposition 3.0.7 (Gluing, general)**.** *Let $\mathfrak{ob}'$ be a relative open books on (the possibly disconnected) manifold M'. Let T_1 and T_2 be two boundary tori of M', oriented as the boundary. $\mathfrak{ob}'$ induces oriented foliations $\mathcal{F}_i$ of T_i, $i = 1, 2$. Let $\psi : T_1 \to T_2$ be any orientation reversing homeomorphism which takes $\mathcal{F}_1$ to $\mathcal{F}_2$. Let $M = M/\psi$ be the manifold formed by identifying T_1 and T_2 via ψ. Let $\mathfrak{ob}$ be the (possibly relative) open book on M formed by joining $\mathfrak{ob}'$. Then there exists a contact structure on M compatible with $\mathfrak{ob}$, unique up to an isotopy fixing the boundary, that restricts* *to a contact structure on M' compatible with $\mathfrak{ob}'$.*

Proof. The construction is straightforward. Since the gluing matches up the oriented pages of $\mathfrak{ob}'$, the open book extends to M. To this, we apply the Thurston-Winkelnkemper construction as before, now additionally requiring the 1-form β on the page Σ of $\mathfrak{ob}$ vanish transversely along the circle in Σ that traces out the torus image of $T_1 \amalg T_2$. This 1-form, then pulls back to a compatible 1-form on M'. □

We point out that one can prove similar compatibility and gluing theorems for relative open books without binding components, that is, for fiber bundles. These can also very useful, for example showing up in the Thurston-Winkelnkemper construction. We will use these in Section 6.2 to construct open book decompositions compatible with the S^1-invariant contact structures on circle bundles. While relative open books with binding can always be equipped with contact structures that are tangent at the boundary tori, fiber bundles can only get close to tangency (though arbitrarily so). This makes it somewhat difficult to determine the Giroux torsion, say, of a contact manifold using an open book decomposition, knowledge that is extremely useful in determining the contact manifold's fillability (c.f. Introduction and Section 4.7.1).

Chapter 4

Open Book Decompositions of torus bundles

4.1 The Classification of Contact Structures on Torus Bundles and $T^2 \times I$.

One of the first classifications of tight contact structures was of those on torus bundles in work done independently by Giroux [14] and Honda [21] [22]. We follow the classification of Honda rather intimately and use much of the terminology presented there. As such, we give a rather cursory overview of the main results. For a good survey of the techniques used, see [13]. The following definitions are from [22].

To any slope s in $\mathbb{R}^2$ associate its standard angle $\bar{\alpha}(s) \in \mathbb{RP}^1 = \mathbb{R}/\pi\mathbb{Z}$. For $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathbb{RP}^1$, let $[\bar{\alpha}_1, \bar{\alpha}_2]$ be the interval $[\alpha_1, \alpha_2] \subset \mathbb{R}$ where $\alpha_i \in \mathbb{R}$ is any lift of $\bar{\alpha}_i$ with $\alpha_1 \leq \alpha_2 < \alpha_1 + \pi$. We say a slope s is *between* s_1 and s_2 if $\bar{\alpha}(s) \in [\bar{\alpha}(s_1), \bar{\alpha}(s_0)]$.

Let ξ be a contact structure on $T^2 \times I$ with convex boundary and assume the dividing Γ_i set on each boundary T_i has two parallel components with slope s_i, $i = 0, 1$. We say ξ is *minimally twisting* if every convex torus $T \times t$ has a dividing set with slope s between s_1 and s_0. For a minimally twisting ξ, the *I-twisting* of ξ is given by $\beta_I = \alpha_1 - \alpha_0$. For a general ξ, cut $(T^2 \times I, \xi)$ into minimally twisting segments $T_k \cong T^2 \times I$, $k = 1, ..., l$ and add up the I-twisting of each: $\beta_I = \beta_{I_1} + \cdots \beta_{I_l}$.

We say a tight contact torus bundle M is *minimally twisting in the S^1-direction* if every decomposition along a convex fiber results in a minimally twisting $T^2 \times I$. Define the *S^1-twisting* β_{S^1} to be the supremum of the I-twisting $\mathrm{Floor}_\pi(\beta_I)$ over all decompositions along convex fibers. Here $\mathrm{Floor}_\pi(x)$ is given by $n\pi$, $n \in \mathbb{Z}$, where $n\pi \leq x < (n+1)\pi$ (in essence, the modulus π floor function).

Theorem 4.1.1 (Theorem 0.1 of [22])**.** *Let M be a T^2-bundle over S^1 with monodromy $A \in \mathrm{SL}(2, \mathbb{Z})$. Then, up to contact isotopy, the tight contact structures are completely classified as in the table below.*

1. *(Universally tight contact structures.) For each A, there exist infinitely many universally tight contact structures, all isotopic as plane fields but distinguished by there S^1-twisting β_{S^1}. Depending on A, the set of possible values for β_{S^1} is $\{2m\pi | m \in \mathbb{Z}^{\geq 0}\}$ or $\{(2m-1)\pi | m \in \mathbb{Z}^+\}$.*

2. *Additionally, there are finitely many others. These are minimally twisting and virtually overtwisted. (There are non-minimally twisting exceptions to this when M is a circle bundle over T^2 with Euler class $e > 1$. There are two, with $\beta_{S^1} = \pi$, which are isotopic only when $e = 2$.)*

Definition 4.1.1 (Basic Slice)**.** A *basic slice* is a tight, minimally twisting contact structure on $T^2 \times I$ with convex boundary. The dividing set on each boundary is composed of two parallel essential curves. The slopes $s_0 = p/q$ and $s_1 = p'/q'$ must satisfy $pq' - p'q = 1$.

Additionally, basic slices are equipped with a sign, $+$ or $-$, which is determined by the relative Euler class of the contact structure. Every basic slice is $\mathrm{SL}(2,\mathbb{Z})$-equivalent to one with slopes $s_1 = 0$ and $s_0 = 1$. These can be constructed by taking the contact structure on $T^2 \times [0,1]$ given by the form

$$\alpha = \sin\left(\frac{\pi}{4}(t-1)\right) dx + \cos\left(\frac{\pi}{4}(t-1)\right) dy,$$

or $-\alpha$ for the basic slice with sign $-$, and perturbing the boundary tori to be convex. According to the classification, all tight contact structures can be constructed by gluing together basic slices. The universally tight family in Theorem 4.1.1 can be constructed using all basic slices of the same sign and whose gluings match up the signs. Under these circumstances, the contact structure given by gluing along convex tori is the same as that given by gluing along prelagrangian tori, provided it is foliated by leaves whose slope is the same as the dividing set of the convex torus.

It is under these circumstances that we may additionally glue the relative open book decompositions. Thus *all* the open books constructed using the relative open books of basic slices will be in the universally tight family of Theorem 4.1.1. To see that we can construct open books compatible with each contact structure in this family, we need to show two things: an example with S^1-twisting $< 2\pi$ and how to add a 'Giroux twist,' i.e. add 2π to β^1_S. We also construct compatible open books for some of the minimally twisting, virtually overtwisted family. In Section 4.6 we construct relative open book decompositions compatible with continued fraction blocks that have at least two positive basic slices. There is still only one way to

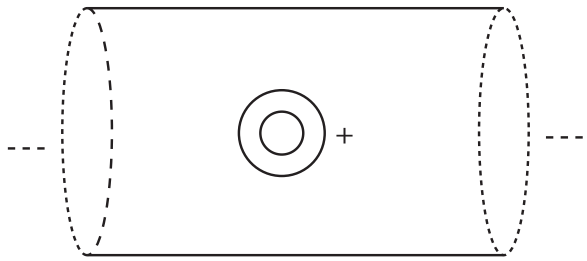


Figure 4.1: Relative open book decompositions for a basic slice.

glue two different continued fraction blocks together, though, and only one way to glue these up to form a (closed) open book. Thus we are unable to construct open books compatible with the very few highly interesting examples of contact structures (those that might be tight but not fillable) using the techniques presented in this paper.

4.2 A Basic Slice

We begin with a prototypical example and show how to get relative open books for any basic slice. To start, we describe an embedding into $T^2 \times I$ of the relative open book decomposition pictured in Figure 4.1 and give a compatible contact structure. Here, the ends of the annulus are boundary circles and trace out the boundary tori in front and back at $t = 0$ and $t = 1$, while the center puncture is a binding circle and will be filled in.

To describe an embedding in $T^2 \times I$, we follow the construction of then ambient manifold from the abstract relative open book. First, form the surface

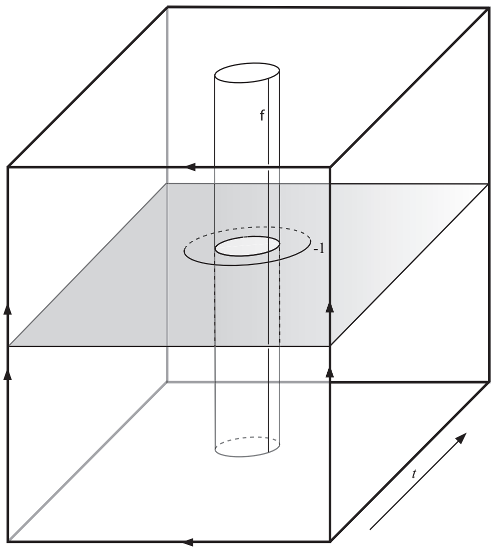


Figure 4.2: An embedded picture of the mapping cylinder for diagram in Figure 4.1.

bundle whose monodromy is given. Here the page Σ is a pair of pants, which we will think of as a punctured annulus. To build the mapping torus, we take $\Sigma \times I$ and glue via ϕ, which is given by a single Dehn twist about a curve γ parallel to the center boundary component. We can describe this by instead taking $\Sigma \times S^1$ and doing -1-framed Dehn surgery (relative to the page) along a copy of γ lying on a fiber. When we fill in the neighborhood of the binding, we do so by the framing traced out by ϕ. In the diagram this is given by a vertical arc f, which is a 0-framed filling using identifications in Figure 4.2.

We may simplify this picture slightly by a procedure analogous to the Rolfsen twist operation on surgery diagrams of 3-manifolds. To apply a Rolfsen twist, we need two components of a surgery diagram: a curve K and a meridian μ, which

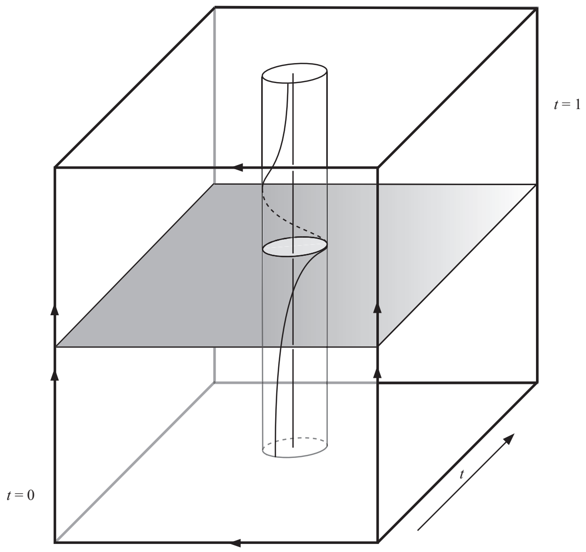


Figure 4.3: Changing the framing curve in Figure 4.2 by a Rolfsen twist.

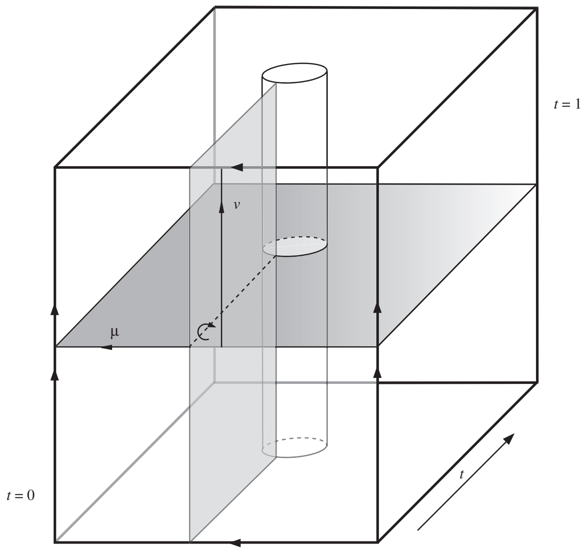


Figure 4.4: A vertical annulus connecting the binding to the boundary torus at $t = 0$.

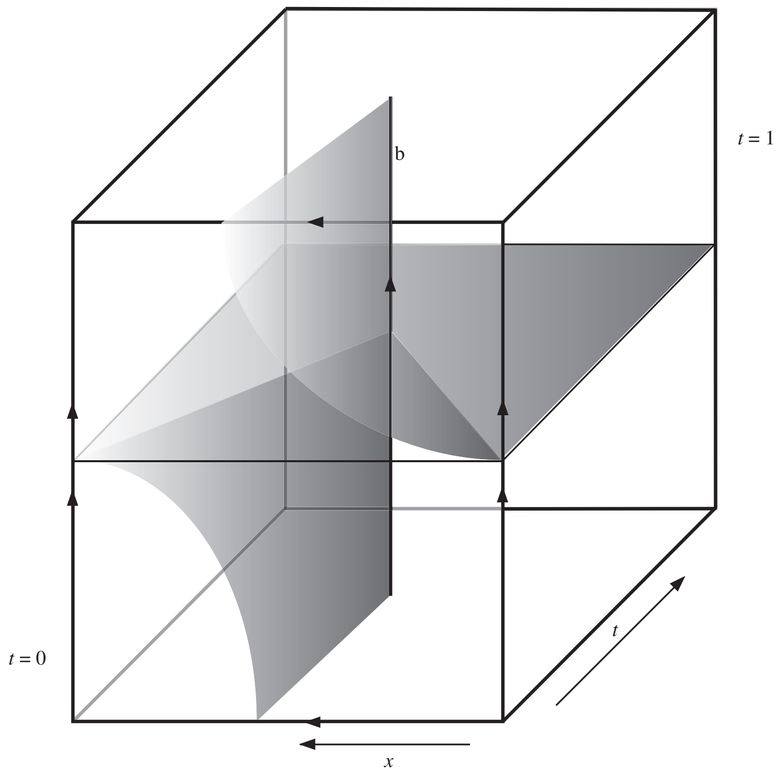


Figure 4.5: New fibration after shearing along the vertical annulus in Figure 4.4 and filling in with the standard fibration of a neighborhood of the binding.

bounds a disk punctured by K. To apply a Dehn surgery along $K \in M$ with framing curve $f \subset \partial\nu(K)$, we cut out a solid torus neighborhood of K and reglue by a map $m : \partial(S^1 \times D^2) \to \partial M$ that sends the meridional disk $p \times \partial D^2$ to f. Removing neighborhoods of μ and K from M, the disk bounded by μ becomes a properly embedded annulus. We may cut along this annulus and twist, simultaneously altering the framing curves for both μ and K, giving a new surgery presentation of the same manifold.

For our purposes, the annulus we cut and shear along will be the annulus component of a page Σ cut along γ (the -1-framed curve in Figure 4.2). If we shear in the correct direction, the -1-framing for γ will become an ∞ framing and the 0-framing f on the vertical solid torus becomes a $+1$ framing, as given in Figure 4.3.

We can take this one step further and shear along a vertical annulus connecting the boundary torus at $t = 0$ to the binding (shown in Figure 4.4). Again, the correct shearing takes the framing curve on the binding to a meridian curve and takes the fibration to that given in Figure 4.5. Here we fill in a neighborhood of the binding by an ∞-filling that takes the longitude λ to the $(-1, 1)$-curve bounding a fiber in the fibration.

Only one fiber is shown in Figure 4.5. To see the rest, translate the fiber vertically.

To construct a compatible contact structure, look at the contact form on

$T^2 \times [0,1]$ discussed before

$$\sin\left(\frac{\pi}{4}(t-1)\right)dx + \cos\left(\frac{\pi}{4}(t-1)\right)dy.$$

One can show the associated contact structure is the isotopic to that of $\alpha = dy + (t-1)dx$. $\ker d\alpha = \partial y$ whose Reeb vector field is everywhere vertical. This vector field is is positively transverse to the fiber, positively tangent to the binding at and vertically (that is, y-) invariant and so exhibits a compatibility between our desired contact structure and our open book. If we perturb the boundary tori to be (minimal) convex, we will have a minimally twisting contact structure with minimal convex boundary and slopes $s_0 = 1$ and $s_1 = 0$, i.e. a *basic slice*.

Both signs of basic slice are isomorphic. We can switch the sign compatible with the open book by changing the orientation of the fibers (and binding) and replacing α with $-\alpha$. However, the difficulty lies in gluing. For contact structures with convex boundary, one can glue using *any* map that preserves the slope of the dividing set. When gluing an open book, you need to match up the oriented fibrations and so only 'half' of the allowed convex gluings are possible using the relative open book decompositions on basic slices given here. In particular, after gluing the relative open book decompositions that we have created the signs of the basic slices agree.

4.3 Exhibiting the Open Books.

The set of all possible open book decompositions on torus bundles obtained by gluing the relative pieces are shown in Figure 4.6. After observing some simple properties and correlating with some difficult work of Honda, we can find a complete set of open books compatible with the family of universally tight contact structures (and some others). The following is proved in Section 4.5. Identify $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Our conventions are that $T_A = \mathbb{R}^2/\mathbb{Z}^2 \times [0,1]/(\mathbf{x},1) \sim (A\mathbf{x},0)$

Theorem 4.3.1. *Let $A \in \mathrm{SL}(2,\mathbb{Z})$ and T_A be the associated torus bundle over S^1. Then each contact structure in the universally tight family of Theorem 4.1.1 can be carried by an open book described by Figure 4.6.*

An immediate consequence is the following theorem (proved in Section 5).

Theorem 4.3.2. *Let $A \in \mathrm{SL}(2,\mathbb{Z})$ and T_A be the associated torus bundle over S^1. Then either (and possibly both) T_A or T_{-A} admits a Stein-fillable contact structure. In particular, the universally tight contact structure on T_A (or T_{-A}) with twisting $\beta^1_S = 0$ is Stein fillable.*

We prove slightly more than that, however. Since any contact manifold with positive Giroux torsion cannot be strongly (and hence Stein) fillable, these examples are the unique Stein fillable and universally tight contact structure on T_A. A

4.4 Preliminary Results and the Relation to the Braid Group

We begin with some notation. Let $Word$ denote the set of words in $\{a, b, a^{-1}, b^{-1}\}$. To each letter we associate the relative open book decomposition shown in Figure 4.7. We should read these diagrams as representing an open book on a manifold with boundary where the interior punctures represent the binding curves and the

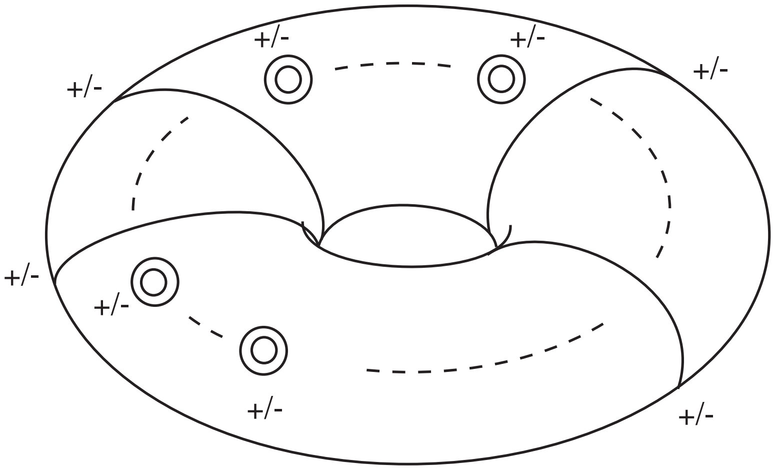


Figure 4.6: General picture for the open book decomposition compatible with ξ_n on a torus bundle. The page is a punctured torus. The monodromy is given as a Dehn multi-twist along the signed curves.

edges of the annulus trace out a pair of torus boundaries. The signed curve on each page segment presents the monodromy of the open book as a Dehn twist.

To any word $w \in Word$, we can then associate an open book with torus pages by stringing together the annular regions associated to each letter in w and identifying the remaining pair of circle boundaries to form a many-punctured torus.

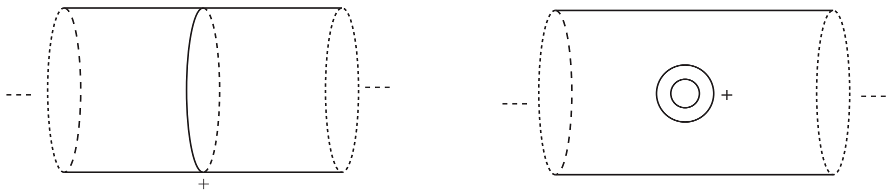


Figure 4.7: Relative open book decompositions for a and b^{-1}. The diagrams for a^{-1} and b are the same with the signs reversed.

The monodromy is given by Dehn twists along the union of all the signed curves. We will denote this open book $\mathfrak{ob}_w = (\Sigma_w, \phi_w)$ and the ambient manifold M_w, where Σ_w is the page of the open book.

Remark 4.4.1. *The diagrams for a^n, $n \in \mathbb{Z}$ are not relative open book decompositions, as they do not have binding components and do not carry a contact structure. One should view the a^n as auxiliary gluing data, describing how the normal directions to the page are matched up when gluing two relative open book decompositions. This poses no problems to our setup and we mention it only for clarification.*

The following lemma describes how two words might describe the same contact structure.

Lemma 4.4.2 (Braid Relation)**.** *Suppose the words $w, v \in Word$ are related by a sequence of braid relations $a^{-1}b^{-1}a^{-1} = b^{-1}a^{-1}b^{-1}$. Then the associated (relative) open books $\mathfrak{ob}_w$ and $\mathfrak{ob}_v$ are stably equivalent.*

Remark 4.4.3. *Before exhibiting a proof, we point out that the relation $aba = bab$ does not hold at the level of contact structures as it involves a negative stabilization. It can thus change the contact structure (and possibly the homotopy type of the plane field). However, any word containing the letter b is not right-veering (indeed, it contains a sobering arc) and so gives an overtwisted contact structure. We will ignore these for the remainder of the paper.*

Proof. The proof itself is straight forward, involving only the lantern relation on the 4-punctured sphere. The braid relation that does hold is $b^{-1}a^{-1}b^{-1} \sim a^{-1}b^{-1}a^{-1}$. We begin with a segment given by $b^{-1}a^{-1}b^{-1}$ as in Figure 4.8.

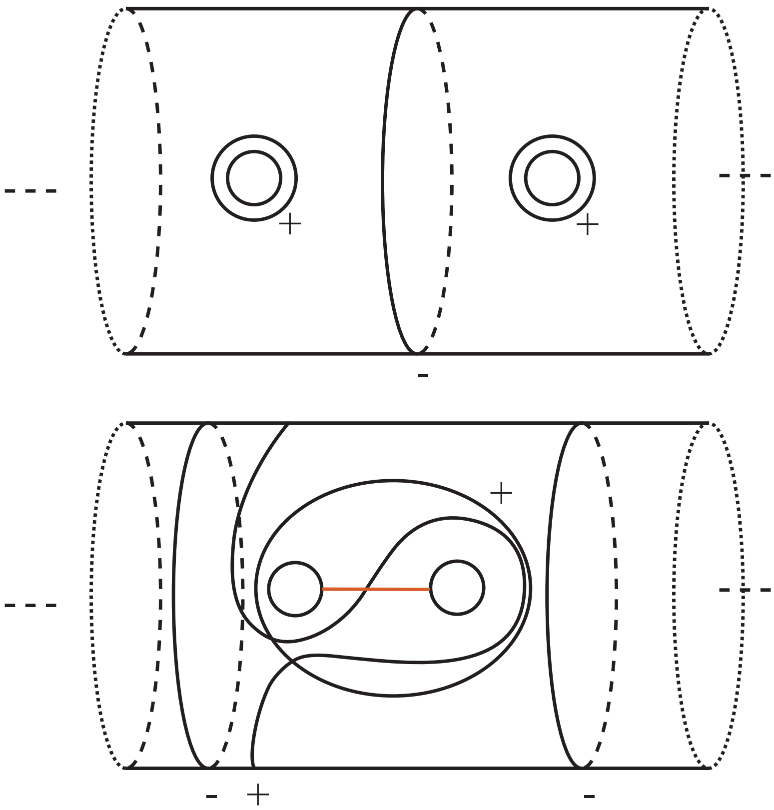


Figure 4.8: Diagrams for $a^{-1}b^{-1}a^{-1}$ before and after the lantern relation. The destabilizing arc is shown.

We think of the page as a 4-punctured sphere and apply the lantern relation to get the diagram on the right. The obvious destabilizing arc is labeled, resulting in the open book described by $a^{-1}b^{-1}a^{-1}$. Notice that this also implies the compatible contact structures are isotopic. While we do not have a complete characterization as in Giroux's Theorem, we do know that Hopf stabilization and destabilization amount to adding or removing a contact (S^3, ξ_{std}), and so do not change the isotopy type of the contact structure.

$\square$

Thus we have shown that the braid relation holds at the level of contact structures via the stabilization/destabilization of open books, giving us a map from equivalent open books to the Braid group. This map descends in a meaning full way to $Aut^+ (T^2)$. There is a natural map

$$\Psi : Word \to Aut^+ \left(T^2\right) \cong \left\langle a, b | aba = bab, (ab)^6 = \mathrm{Id} \right\rangle \cong \mathrm{SL}(2, \mathbb{Z})$$

and we often abuse notation and use $\Psi(w)$ to denote the automorphism and the associated element of $\mathrm{SL}(2, \mathbb{Z})$ with the assumption we've identified T^2 with $\mathbb{R}^2/\mathbb{Z}^2$. We will show that $M_w = T^2 \times_{\Psi(w)} S^1$.

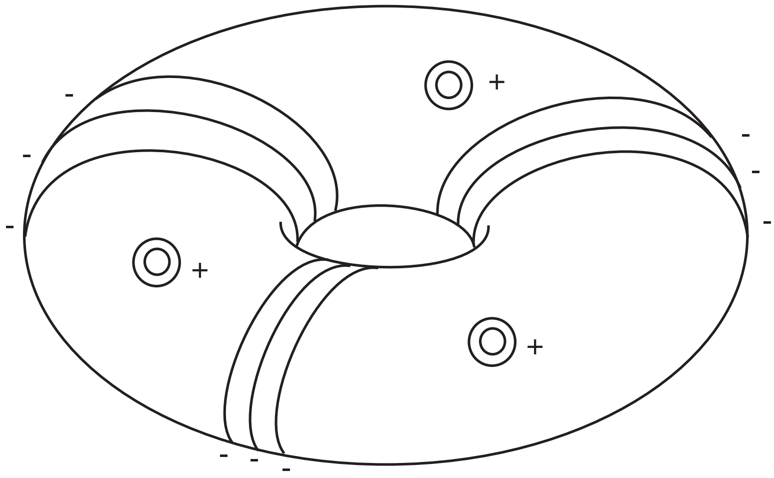
Lemma 4.4.4. *Let M be the ambient manifold of the open book $\mathfrak{ob}_w$. Then M is homeomorphic to the torus bundle $T_A \cong T^2 \times_{\Psi(w)} S^1$, whose monodromy is given by $A = \Psi(w)$.*

Proof. The proof follows directly from the cut-and-paste construction of the open book $\mathfrak{ob}_w$, building the surface bundle $\Sigma_w \times_{\phi_w} S^1$ and filling in near the binding. We point out the following fact which will be used in the construction of the mapping torus.

Fact: Let γ be a curve embedded on a fiber F of a surface bundle M. Then the manifold obtained by Dehn surgery on γ with framing $pf - 1$ (resp. $pf + 1$) is isomorphic to the surface bundle obtained by cutting M along F and regluing by a positive (resp. negative) Dehn twist D_γ along γ. Equivalently, if ϕ is the monodromy of M measured by a return map to F, then the monodromy of the surgered manifold is $\phi \circ D_\gamma$. (Here, pf is the framing of γ given by a pushoff along the fiber F.)

For our purposes, ϕ_w is a multitwist along a union of curves Γ, and so we can form $\Sigma_w \times_{\phi_w} S^1$ by doing Dehn surgery along the image of Γ in $\Sigma_w \times \{pt.\} \subset \Sigma_w \times S^1$. In order to more easily picture this we identify $\Sigma_w \times S^1$ with $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ with vertical tubes drilled out for each of the binding components (that is, the fibers of the bundle are horizontal slices) and Γ sitting on $S^1 \times S^1 \times \{1/2\}$. Topologically, the binding glues in to form a 0-framed Dehn filling, which we view as 0-framed Dehn surgery along the cores of the vertical tubes. Blowing down the components of Γ which are meridians to the vertical tubes (now 0-framed surgery curves) gives a surgery presentation of M composed of ±1-Dehn surgeries in T^3 along curves that lie on *vertical tori* $\{pt.\} \times S^1 \times S^1$. a and b correspond to -1-surgeries, while a^{-1} and b^{-1} give $+1$-surgeries. The page framings are the same as that of the vertical tori and hence give a monodromy presentation of a torus bundle over S^1 corresponding to $\Psi(w)$.

$\square$



Proof.

Figure 4.9: An Open Book Decomposition for the Stein Fillable Contact Structure on T^3.

Corollary 1. *Any word in* $\{a, a^{-1}, b^{-1}\}$ *gives an open book decomposition corresponding to a weakly-fillable contact structure.*

We begin by showing the open book given by $w = (a^3b)^{-3}$ corresponds to the unique Stein-fillable contact structure ξ_0 on T^3. By Lemma 4.4.4, the ambient manifold M_w is T^3. To see this is Stein fillable, we use only the *star relation* of [10] to present ϕ_w as a product of positive Dehn twists (conjugates of the standard generators). (In fact, this does more. It gives a Lefschetz fibration on $T^2 \times D^2$ exhibiting a Stein structure.) Thus any cover of $(M_w, \mathfrak{ob}_w)$ gives a weakly-fillable open book decomposition. Take the cover that gives $(a^3b)^{-3n}$. (We will see later that these have Giroux torsion ≥ 1 and so cannot be Strongly fillable.) Now take an arbitrary word v in $\{a, a^{-1}, b^{-1}\}$, which we will consider up to cyclic permutation and canceling adjacent aa^{-1} pairs. We will use the braid relation to ensure that the letter a^{-1} occurs at most three times in sequence. Assume we have the subword $b^{-1}a^{-n}b^{-1} \subset v$. Since $(a^{-1}b^{-1}a^{-1})a^{-1} \sim b^{-1}(a^{-1}b^{-1}a^{-1})$, we can write $b^{-1}a^{-n}b^{-1} \sim aa^{-1}b^{-1}a^{-1}a^{-(n-1)}b^{-1} \sim ab^{-(n-1)}a^{-1}b^{-1}a^{-1}b^{-1}$ and thus reduce the length of any string of a^{-1}s. Lastly, notice we may insert the letter a into any word via Legendrian surgery. Thus, after possibly modifying v using the braid relation as above (and ensuring that b^{-1} occurs some multiple of 3 times) of we can realize it via Legendrian surgery on the weakly-fillable contact manifold given by $(a^3b)^{-3n}$, showing it is weakly-fillable. □

4.5 Construction of Minimal Examples

We prove Theorem 4.3.1 here. We follow Lemma 2.1 of [22] use $\mathrm{SL}(2,\mathbb{Z})$ to find easy matrix representatives for each torus bundle, and their factorizations (this data is summed up nicely in the tables following Theorem 0.1 of [22]). Using this, we can find words in $Word$ that exhibit universally tight contact structures with *small twisting* in the S^1-direction, i.e. $\beta_{S^1} < 2\pi$. We can then produce every contact structure by adding $(aba)^{-4}$ segments to our small twisting examples, increasing β_{S^1}. To measure the twisting, we use the word to cut the torus bundle into $T^2 \times I$, thinking of $t = 0$ as an incoming boundary and $t = 1$ as an outgoing one. The pages have an angle on the vertical tori (and are given by a primitive vector in $\mathbb{Z}^2$ or its angle $\theta \in [0, 2\pi)$), not just a slope, and we use $\mathrm{SL}(2,\mathbb{Z})$ to assume the fiber at $t = 1$ has angle $(1, 0)$. For ease, we orient the angle at $t = 0$ the wrong way, that is, as an incoming boundary of the page and not by the outward normal. We can then measure the twisting by following the progression of the angles of the pages.

(In one sense, this is an oriented version of examining the twisting of a convex decomposition by looking at the progression of the slopes of the dividing sets.) To that end, identify $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and let $b^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Moving from $t = 1$ to $t = 0$, after crossing each basic slice the angle of the pages of the open book changes and the contact planes rotate along with it. For example, suppose $w = a^{k_0}b^{-1}a^{k_2}b^{-1}\cdots a^{k_{n-1}}b^{-1}a^{k_n}$. Which we read from left to right as we move from $t = 0$ to $t = 1$. To compute the angles of the pages we end with $s_n = (1,0)$ and work backwards towards $t = 0$. Then $s_{n-1} = a^{k_{n-1}}b^{-1}(1,0)$ and similarly $s_{n-i} = a^{k_{n-i}}\cdots b^{-1}a^{k_n}b^{-1}$ and so forth. To begin, let $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = bab, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = a$. We will abuse notation and associate to every word w its image in $\mathrm{SL}(2,\mathbb{Z})$ under Ψ. Conjugacy classes of elements of $\mathrm{SL}(2,\mathbb{Z})$ can be divided into three groups depending on $\mathrm{Tr}(A)$. We enumerate the possibilities below:

Proof. Proof of Theorem 4.3.1

1. Elliptic: $|\mathrm{Tr}(A)| < 2$

   (a) $A = -S$ A is a rotation by $\pi/2$. We factor $A = a^{-1}b^{-1}a^{-1}$. This is a decomposition into a single basic slice and so has small twisting and $\beta_S^1 = 0$.

   (b) $A = S$ A is a rotation by $-\pi/2$. We factor $A = (a^{-1}b^{-1}a^{-1})^3$. This decomposition has 3 basic slices, with boundary angles $s_3 = (1,0)$, $s_2 = (0,1)$, $s_1 = (-1,0)$, $s_0 = (0,-1)$. It has small twisting. $\beta_S^1 = \pi$.

(c) $A = -T^{-1}S$. We factor: $A = a^{-2}b^{-1}a^{-1}$. Again, this has a single basic slice. $\beta^1_S = 0$.

(d) $A = -(T^{-1}S)^2$. We factor: $A = a^{-1}b^{-1}$. This is a single basic slice and so $\beta^1_S = 0$.

(e) $A = T^{-1}S$. We factor: $A = (ba)^{-5}$. Each $(ba)^{-1}$ is the $\mathrm{SL}(2,\mathbb{Z})$ equivalent of a rotation by $\pi/3$, so the decomposition into 5 basic slices has small twisting and $\beta^1_S = \pi$.

(f) $A = T^2 - 1S$. We factor: $A = (ba)^{-4}$. This has small twisting by the previous argument and $\beta^1_S = \pi$

2. Hyperbolic: $|\mathrm{Tr}(A)| > 2$

(a) $A = T^{r_0}ST^{r_1}S \cdots T^{r_k}S$, where $r_0 < -2$ and $r_i \leq -2$. We can rewrite this as $A = a^{r_0}(aba)a^{r_1}(aba)\cdots a^{r_k}(aba) \cong a^{r_0+2}ba^{r_1+2}\cdots a^{r_k+2}b$. Now we conjugate again to swap a and b to get $A = b^{r'_0}ab^{r'_1}a \cdots b^{r'_k}a$ with $r'_i = r_i + 2 \leq 0$ and $r'_0 < 0$. Since, the exponent for b is always negative, this word gives a weakly fillable contact structure. Further, notice that if w is any word composed of the letters a and b^{-1} (that is, there are no a^{-1}s involved), then $\Psi(w)$ is a matrix with positive entries and so all of the angles s_i lie between 0 and $\pi/2$. This gives an open book with small twisting and $\beta^1_S = 0$.

(b) $A = -T^{r_0}ST^{r_1}S \cdots T^{r_k}S$, where $r_0 < -2$ and $r_i \leq -2$. Again, we choose the factorization above, but with the sign reversed. This gives a word $A = (aba)^{-2}b^{r'_0}ab^{r'_1}a \cdots b^{r'_k}a$ with $r'_i = r_i + 2 \leq 0$ and $r'_0 < 0$. The total twisting in this case is less than $3\pi/2$ and $\beta^1_s = \pi$.

3. Parabolic: $|\mathrm{Tr}(A)| = 2$. $A = \pm T^n$, $n \in \mathbb{Z}$. Cases:

   (a) $A = T^n$, $n > 0$. These are the circle bundles over T^2 with Euler number n. The illegal factorization is $A = a^n$, which we alter to $A = (aba)^{-4}a^n$. This word has small twisting (indeed it is given as Legendrian surgery on (T^3, ξ_0)). Technically, these examples have $\beta_{S^1} = 2\pi$ (see footnote).

   (b) $A = Id$. This is T^3. The factorizations $A = (aba)^{-4} \cong (a^3b)^{-1}$ give ξ_0, the unique Stein fillable contact structure on T^3, which was shown in the proof of Corollary 1 in Section 4.4. We will see this in more detail later. Again, an odd case with $\beta_{S^1} = 2\pi$ (see footnote).

   (c) $A = T^n$, $n < 0$. While the immediate factorization is $A = a^n$, this word doesn't correspond to an open book. Conjugate to $A = b^n$, since $n < 0$, to get an open book with angles $s_k = (n-k, 1)$. This has small twisting and $\beta_{S^1} = 0$.

   (d) $A = -T^n$, $n \in \mathbb{Z}$. The appropriate factorizations here are $A = (aba)^{-2}a^n$. The decomposition is into two basic slices, with $\beta_{S^1} = 0$ if $n > 0$ and $\beta^1_S = \pi$ if $n \leq 0$.

Footnote: While these technically have $\beta_{S^1} = 2\pi$, Honda [22] lists these as having $\beta_{S^1} = 0$. By the vagaries of terminology, these examples still have Giroux torsion equal to 0 (since Giroux torsion measures an embedding of a closed interval of full twisting into the contact manifold). Both examples have positive representatives for the monodromy and so are Stein fillable.

□

4.6 Virtually Overtwisted Continued Fraction Blocks

In addition to the basic slices used in constructing the universally tight contact structures on torus bundles, we have examples of relative open books for *continued fraction blocks* on $T^2 \times I$. As long as there are at least two positive basic slices, we can build compatible relative open books that have many negative basic slices hidden within them. Continued fraction blocks, in essence, are strings of equivalent basic slices; one can rearrange their order (swapping negative for positive and so on) without changing the isotopy type. A continued fraction block with p positive basic slices would be carried by the relative open book b^{-p}. We'll build more with k negative and $p-k$ positive basic slices, $p-k \geq 2$. We first show how to construct the examples, following a surgery description discovered by Schönenberger [31]. We then mimic Honda's classification, completing the open books on Lens spaces whose compatible contact structures are distinguished by the Chern classes of the Stein fillings, following results of Lisca and Matić [25]. Since there is a unique way to complete each contact structure and each completion is distinct, we know each of the basic slices is distinct. Further, since we can determine which contact structure on $L(p,1)$ we complete to, we deduce the decomposition of the continued fraction block into positive and negative basic slices.

Schönenberger [31] gave examples of open book decompositions for all

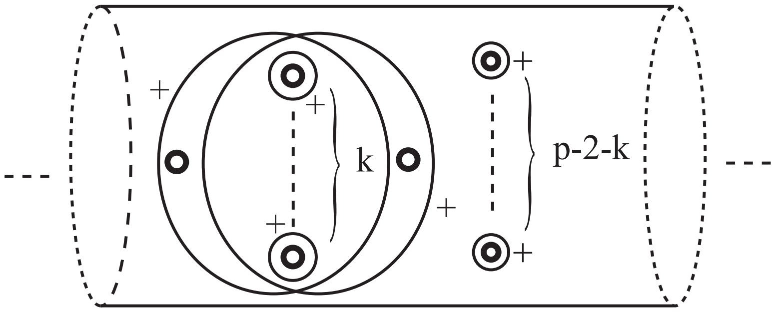


Figure 4.10: Continued fraction blocks $CF(p, k)$ on $T^2 \times I$ with p basic slices, $p - k$ positive and k negative. The page is a punctured annulus with two boundary circles. The bold circles are the binding.

contact structures on Lens spaces by finding nice Legendrian surgery diagrams. The idea is to *roll up* the surgery diagram so that each successive surgery is done on a stabilization of a pushoff of the previous. Since we need only a single continued fraction block, we will describe the procedure for only the easiest diagrams (those obtained by a single Legendrian surgery on the unknot) and show how to extract the appropriate $T^2 \times I$ segment from it. The continued fraction blocks we will look at are described in Figure 4.10. Each requires at least two positive basic slices, which we can think of as hugging the negative basic slices, ensuring the contact planes all glue with the correct orientation when we stack the continued fraction blocks together. When $k = 0$, all the basic slices are positive and we have the diagram b^{-p} constructed before. A proof that the ambient manifold is indeed $T^2 \times I$ follows in Section 4.6.1.

To distinguish the different examples, we embed each into an open book for $L(p+2, 1)$, by capping off each boundary circle with the open book in Figure 4.11.

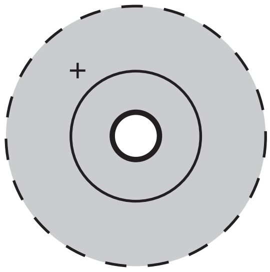


Figure 4.11: A relative open book on $S^1 \times D^2$ used to cap off the continued fraction blocks.

This gives the standard neighborhood of the binding (in $S^1 \times D^2$) and we fill by a map that glues the pages together. If we perturb the prelagrangian gluing torus to become convex, the dividing curves have slope parallel to the pages, and so looking on the boundary of the solid torus, Γ has slope 0. Thus, there is a unique way to extend ξ to a tight contact structure over (each) $S^1 \times D^2$. The capped of diagrams are shown in Figure 4.12.

Each of the tight contact structures on $L(p, 1)$ is given by Legendrian surgery on the unknot. We will use this description to determine which contact structures they represent and use this data to distinguish them.

To describe the open book decompositions, we begin with the unknot, Legendrian realized on a page of the open book given by the positive Hopf link H^+. The page and contact framings are both -1 and so Legendrian surgery gives a contact structure on $L(2, 1)$ compatible with the open book shown in Figure 4.14.

To exhibit the contact structures for $p > 2$ we add left and right zig-zags to the front projection (up and down Legendrian stabilizations), giving the diagrams

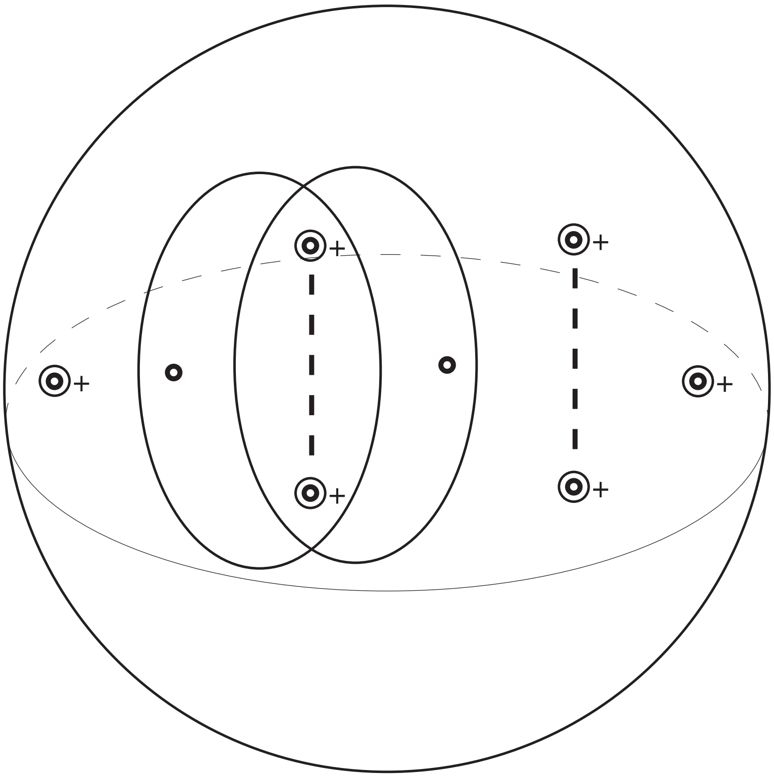


Figure 4.12: Continued fraction blocks embedded into $L(p, 1)$. The page is a sphere. The bold circles punctures representing the binding.

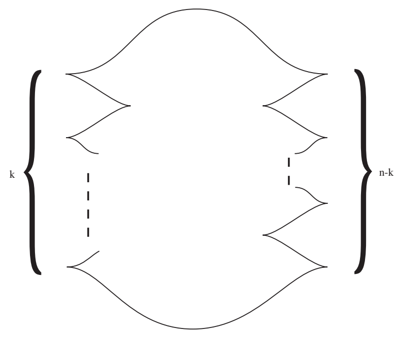


Figure 4.13: Legendrian surgery diagrams for distinct contact structures on $L(p, 1)$.

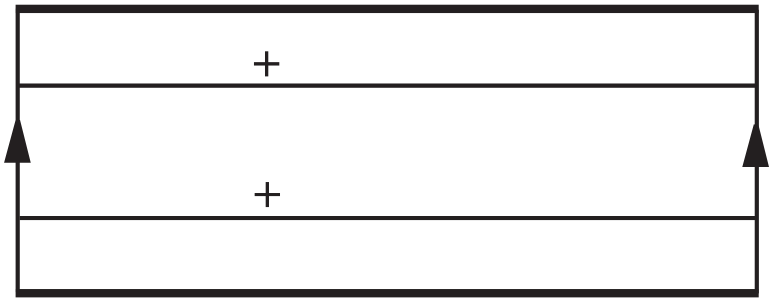


Figure 4.14: An open book decomposition with annular page for the unique tight contact structure on $L(2,1)$ (the left and right boundaries are identified).

shown in Figure 4.13. We can add zig-zags to an open book decomposition as well. By stabilizing the open book k times along one component of H^+ and $(p+2)-k$-times along the other, we can Legendrian realize any of the different stabilizations on a page of an open book for S^3. An example of such a stabilization is shown in Figure 4.15. Removing one of the Dehn twists along a curve that traverses the annulus leaves an open book for S^3. Such a curve is a stabilization of the standard unknot.

Lisca and Matić [25] showed the diagrams in Figure 4.13 give non-isotopic contact structures for different k (these can be distinguished by the Chern classes of the Stein fillings given by Legendrian surgery) and so the open books in Figure 4.12 are compatible with distinct contact structures. Since there was a unique way to complete the continued fraction block, each must also be distinct.

These pieces allow us to construct 'most' of the remaining open book decompositions on the circle bundles $A = T^n$ and the positive hyperbolic bundles. As an example, we give an open book on the hyperbolic torus bundle with $A =$

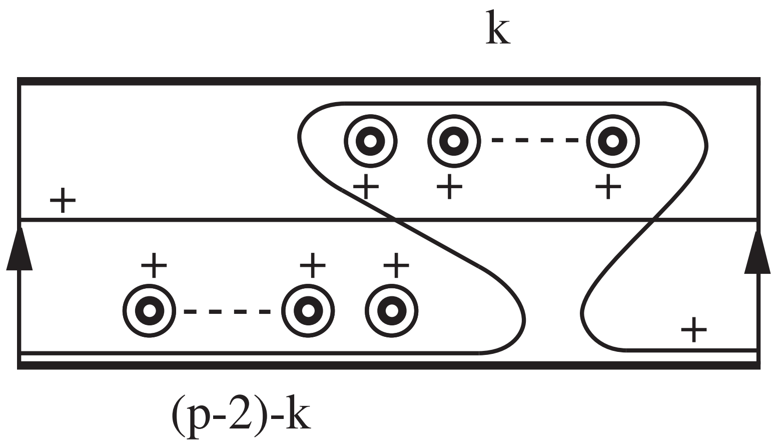


Figure 4.15: Planar open books for the tight (Stein-fillable) contact structure on on $L(p, 1)$.

$b^{-4}ab^{0}ab^{-5}ab^{0}a$ compatible with a virtually overtwisted contact structure in Figure 4.16.

4.6.1 Another Surgery Description

It is fairly straightforward, using the surgery description of the ambient manifold provided by an (relative) open book, to see the open book decompositions $CF(p, k)$ are indeed of $T^2 \times I$. In order to illustrate more clearly the behavior of the embeddings of these open books, however, we will proceed in a slightly more detailed way. We build up the surgery description of Figure 4.10 and show it gives $T^2 \times I$. Since we already know the $b^{-(p-k-2)}$ segment gives a $T^2 \times I$, we assume that $k = p - 2$.

Without the dotted circle in the monodromy, the open book is a stabilization of the relative open book $(\mathbb{A}, Id)$, with page a punctured annulus, where the

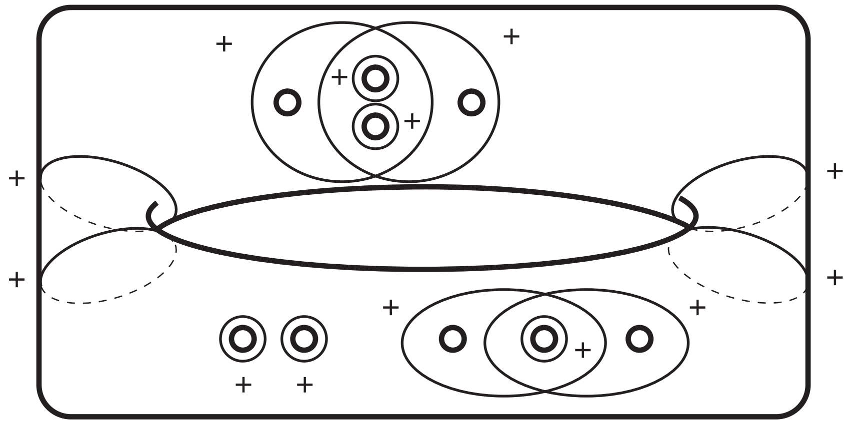


Figure 4.16: An open book compatible with a (Stein-fillable) virtually overtwisted contact structure on the torus bundle with $A = b^{-4}ab^0ab^{-5}ab^0a$.

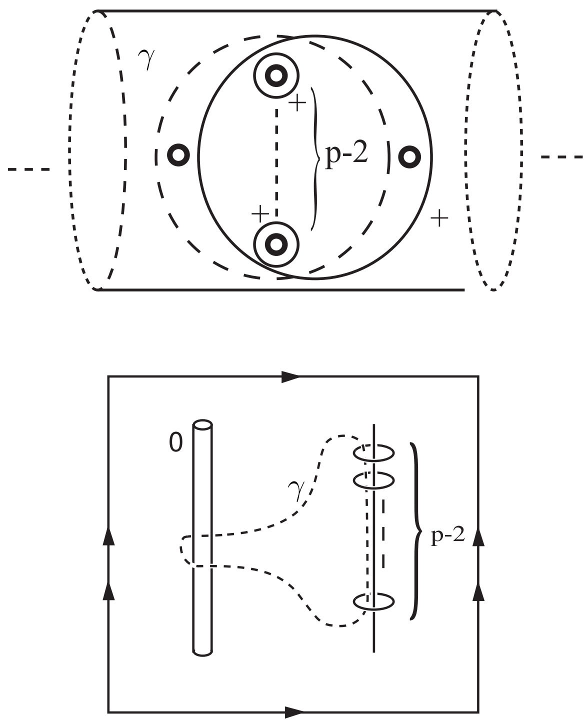


Figure 4.17: Relative open book and embedded picture beginning the description of the continued fraction block $CF(p)$.

puncture is a binding circle b and the boundary components of $\mathbb{A}$ are boundary circles (the manifold is $S^1 \times D^2 \# S^1 \times D^2$ given by 0-surgery on a vertical curve in $T^2 \times I$). The first stabilization is off b and adds a meridian μ of b to the binding, which becomes a longitude in the surgery picture on $T^2 \times I$. We can see the embedded fibration by looking at the fibration for a basic slice (c.f. Figure 4.5) and adding a 0-surgery on a vertical curve. The second $p-2$ stabilizations are off μ, and add $p-2$ meridians of μ to the binding. We can see an embedded description of this in Figure 4.17. The diagram depicts $T^2 \times I$ with its 0-framed vertical curve and the rest of the binding obtained by stabilizations of its core. We can see the fibration by looking again at the basic slice of Figure 4.5 (whose binding is μ) adding $p-2$ meridians to this binding and a 0-framed vertical curve. The $p-2$ meridians alter the fibration only in a neighborhood of μ and only by adding twisted bands to the original fibers. The curve γ is a stabilization of a parallel copy of b, one positive and $p-2$ negative. In particular, the framing of γ given by the page is $1-p$ and so Legendrian surgery is smooth $-p$-surgery. Ignoring now the fibration, γ is a meridian to the 0-framed vertical curve. A slam dunk changes the surgery picture to a $1/p$-framed vertical curve, which can again be Rolfsen twisted to the boundary. While it is non-trivial to follow the fibration through these last two steps, the surgery description does indeed give $T^2 \times I$.

4.7 Some Examples

4.7.1 Weakly but not Strongly Fillable

Recent work of David Gay [9] proves a conjecture of Eliashberg: any contact structure with nontrivial Giroux torsion cannot be strongly fillable. This is an extremely useful in determining the fillability of a contact structure. In particular, of all the universally tight open books on torus bundles, only one has non-zero Giroux torsion, and so the rest cannot be strongly fillable (though a weak filling is easy to construct by taking the appropriate Lefschetz fibration over D^2). The relative open book $(aba)^{-4}$ is a $T^2 \times I$ segment with non-trivial Giroux torsion. This is not to say that any open book decomposition containing a region isomorphic to $(aba)^{-1}$ has non-trivial Giroux torsion, however. The compatible contact structure on $(aba)^{-1}$ includes only when $(aba)^{-1}$ is glued to another *relative* open book. As long as there is another binding component somewhere, then we can conclude that $\mathfrak{ob}$ has non-trivial Giroux torsion. If $(aba)^{-1}$ is instead glued to a fiber bundle (c.f Section 6.2), the contact structure might be Stein fillable, as is the case for T^3.

4.7.2 Strongly but not Stein Fillable

The first examples of strongly fillable contact structures that cannot support a Stein filling were constructed by Ghiggini [11]. We these are given by Legendrian surgery along a curve in the contact structures compatible with $w_n = (a^{-1}b^{-1})(aba)^{-4n}$. The Legendrian curve an be realized on a page of this open book decomposition. The resulting open book is given in Figure 4.18.

Without the Dehn twist along the longitude added to the monodromy, this

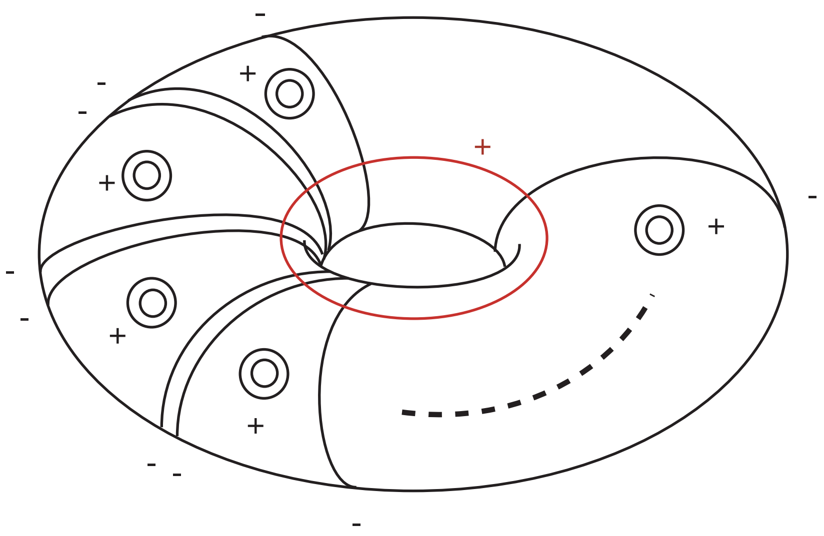


Figure 4.18: Open book decomposition for a Strongly fillable contact structure that doesn't admit a Stein filling.

open book is only weakly fillable. After the surgery, the manifold becomes a homology sphere and so any weak-filling can be perturbed into a strong filling. Some clever manipulations of Heegaard Floer Homology are used to show the contact structure cannot be Stein fillable.

4.7.3 An Interesting Example with Zero Giroux Torsion

There is one interesting family of non-examples in Theorem 5.0.3, namely the circle bundles over the Klein bottle $A = -T^n$ which we will write as $(aba)^{-2}a^n = a^{-2}b^{-1}a^{n-2}b^{-1} = w_n$. For $n > 0$ these are Stein fillable (use the 2-chain relation), but for $n << 0$ these do not have monodromies in $Dehn^+$, as follows. Suppose $\phi_{w_n} \in Dehn^+$ for some $n < 0$. This gives a factorization $\phi_{w_n} = \partial_1\partial_2 a_1^{-2} a_2^{n-2} =$

$\gamma_1 \cdots \gamma_k$ where each γ_i is a right-handed Dehn twist on the twice punctured torus Σ. Rearranging gives a presentation $\partial_1 \partial_2 = \gamma_1 \cdots \gamma_k a_1^2 a_2^{-n+2}$. Any such presentation gives an elliptic fibration over S^2 with 2 sections of square -1 and $-n+4+k$ singular fibers. However, the classification of elliptic fibrations is known: if the number of singular fibers is $12m$, the fibration is $E(m)$ which is not minimal for $n > 1$, therefore $n > -8$. Notice, though, that we can apply the braid relation to get $w_n \cong w_n' = b^{-2}a^{-1}b^{n-2}a^{-1}$. While this is very close to being in $Dehn^+$, it too cannot be written as a product of Dehn twists for $n < -5$, since no elliptic fibration admits more than 9 disjoint sections of square -1 (and so $n > -5$). Either these give examples of fillable contact structures with Giroux torsion $Tor = 0$ that do not admit a Stein filling, or this gives an example where one can destabilize out of $Dehn^+$. There is still a candidate Stein fillable contact structure (T_A admits a Stein filling) given by taking the minimally twisting contact structures on T_A with $A = T^n$, cutting along a minimal convex torus and regluing by a rotation by π. We cannot construct compatible open books for these contact structures using the techniques presented here, however. It is true (though outside the scope of this paper) that these contact structures are Stein fillable, which can be seen by looking at the right surgery diagram.

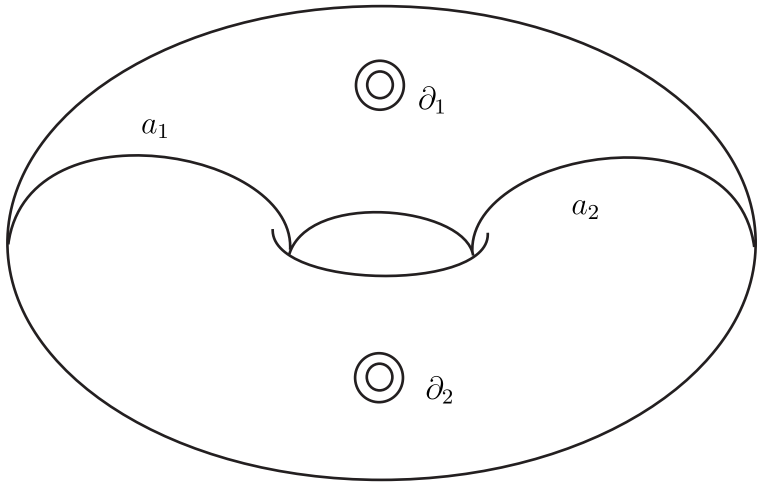


Figure 4.19: Curves on the twice punctured torus generating ϕ_{w_n}.

Chapter 5

Stein Fillings

There are many equivalent definitions of a Stein manifold in $4-dimensions$. In [17], Gompf details a handle calculus for Stein 4-manifolds utilizing work of Eliashberg [3] and Weinstein [35]. The 0-handle is the standard symplectic 4-ball with convex contact boundary S^3. Symplectic 1- and 2-handles can then be added, requiring that the 2-handles are attached along Legendrian knots with the appropriate framing. After every stage the boundary 3-manifold inherits a contact structure. This geometric picture is paired with a more topological description by Loi and Piergallini in [26], where the Stein condition of [17] is shown to be equivalent to a *positive allowable Lefschetz fibration*, or PALF. While this relation isn't as strong as the Giroux correspondence (nor does it cover all symplectic/contact manifolds), it is a good 4-dimensional parallel. It is in this language that we construct our Stein fillings.

A *Lefschetz fibration* on a 4-manifold X is a singular fiber bundle $\pi : X \to \Sigma$, that is there are finitely many critical points $\{x_i\} \subset X$ of π and away from the fibers containing the critical points, π is a fiber bundle. Each fiber containing a critical point $\pi^{-1}(\pi(x_i))$ is an immersed surface called a *singular fiber*. π has a local chart near each critical point, given (in complex coordinates) by $\pi(z_1, z_2) = z_1^2 + z_2^2$.

If the complex orientation agrees with that of X then the critical point is called a *positive singularity* and otherwise it is called a *negative singularity*. Each singular fiber has a natural local description. Let's suppose the only critical point in the singular fiber F is p_i. Taking any arc $\gamma \in \Sigma$ from F to a nearby smooth fiber $\pi^{-1}(q)$, we may look at the totally real disk lying in the local chart around p_i, containing p_i and lying above the arc γ. This disk is called the *thimble* of p_i and its boundary in $\pi^{-1}(q)$ is called the *vanishing cycle* v of p_i. The singular fiber F can be obtained from a smooth fiber by collapsing v. Further, let X_{p_i} be the preimage under π of a small neighborhood in Σ of γ. $\pi^{-1}(\partial X_{p_i})$ is a fiber bundle over S^1. The fiber is that of π and the monodromy is given by a single Dehn twist about v, positive (resp. negative) if p_i is a positive (resp. negative) critical point of π. A critical value is *allowable* if the associated vanishing cycle is homologically essential in $H_1(\Sigma)$.

Theorem 5.0.1. *Loi-Piergallini [26] A Stein surface is equivalent, up to orientation preserving diffeomorphism, to a positive allowable Lefschetz fibration with bounded fiber and base D^2.*

The proof involves taking a Legendrian handle diagram for the Stein surface and showing (by explicit manipulation of the diagram) that it corresponds to a branched cover of $B^2 \times B^2$ along a positive braided surface (in the terminology of Rudolph this is a 'quasipositive surface') and hence to a PALF. Indeed, the boundary of a positive braided surface is a quasipositive braid in S^3. (A braid is *quasipositive* if its braid word is a product of conjugates of positive half twists.) We can make this braid transverse to the trivial open book on S^3 (with binding the braid axis) and branching along this braid gives an open book decomposition on the cover whose monodromy is explicitly positive. Most importantly, all this can be done so that the Stein structure supported by the PALF induces a contact structure on the boundary that is carried by the open book decomposition.

This is an example of a more general topological observation. Let (X, π) be a PALF with bounded fiber Σ and base D^2. Then $\partial(X, \pi)$ is an open book decomposition of ∂X. This is immediate from our abstract construction of an open book decomposition. $\partial(X, \pi)$ has two components, the mapping torus $\pi|_{\partial D^2}$, and the solid tori $\partial\Sigma \times D^2$. These two components are glued together so that boundary of each disk in $\partial\Sigma \times D^2$ gets glued to $.pt \times S^1$ in the mapping torus and so creates an open book decomposition.

Let $Dehn^+$ denote the monoid in $Aut^+(\Sigma)$ generated by (isotopy classes of) positive Dehn twists.

Corollary 5.0.2. *A contact manifold (M, ξ) is Stein fillable iff it is compatible with an open book decomposition with positive monodromy, $\phi \in Dehn^+$.*

Proof. We have seen that the boundary of any PALF over the disk inherits an open book with positive allowable monodromy. Further, the induced contact structure and open book are compatible. The other implication is just as obvious but with one thing to note here: any positive monodromy can be made into an *allowable* positive monodromy. Such an alteration is straightforward. Replace any Dehn twist about a null-homologous curve γ with a chain relation on the punctured surface

$\subset \Sigma$ bounded by γ. (Note, this can be done in many ways, and different relations may give different Stein fillings.) $\square$

Theorem 5.0.3. *Let $A \in \mathrm{SL}(2, \mathbb{Z})$ and T_A be the associated torus bundle over S^1. Then either (and possibly both) T_A or T_{-A} admits a Stein fillable contact structure. In particular, there is a unique universally tight, Stein fillable contact structure on either T_A or T_{-A}.*

This is essentially a corollary of the previous section, using the factorizations given above. In particular, the factorizations used give monodromies that can be written as a product of positive Dehn twists. The uniqueness We will use the following relations in the braid group.

Proposition 5.0.4. *The following relations hold in the mapping class group of a bounded surface*

1. *The chain relations ([34]). A* chain *of curves on a orientable surface is a collection of smoothly embedded curves $\{\gamma_i\}_{i=1}^n$ such that each γ_i intersects only the curves γ_{i-1} and γ_{i+1} and only transversely and at a single point (set $\gamma_0 = \gamma_{n+1} = \emptyset$). Then an interval neighborhood of $\cup\gamma_i$ is a compact surface Σ. If n is even, Σ has a single boundary component. Let δ be a positive Dehn twist about $\partial\Sigma$ and D_i a positive Dehn twist about γ_i. Then the following holds in the mapping class group of Σ:*

$$\delta = (D_1 D_2 \cdots D_n)^{(2n+2)}.$$

If n is odd, Σ has a two boundary components. Let δ be a positive Dehn multitwist about $\partial\Sigma$ and D_i a positive Dehn twist about γ_i. Then the following holds in the mapping class group of Σ:

$$\delta = (D_1 D_2 \cdots D_n)^{(n+1)}.$$

2. *We want to point out that the word $(D_1 D_2)^6$ on the punctured torus is braid equivalent to both $(D_1 D_2 D_1)^4$ and $(D_1^3 D_2)^3$, giving two additional (equivalent) relations on the punctured torus.*

3. *The star relation [10] (c.f proof of Corollary 1). Let Σ be the three-punctured torus and α_i, $i = 1, 2, 3$ and β be the curves on Σ as given by Figure 5.1. Let δ be the positive Dehn multitwist about $\partial\Sigma$, D_i the Dehn twist about α_i and D_β that about β. The following holds in the mapping class group of Σ:*

$$\delta = (D_1 D_2 D_3 D_\beta)^3$$

Recalling the factorizations of Section 4.5, we construct as many positive open books as we can. First some conventions. As before, to any word $w \in Word$ we can associate an open book decomposition of the torus bundle $T_{\Psi(w)}$. The page will be a punctured torus, denoted Σ and the monodromy will be denoted by ϕ. ϕ will always have a presentation in Dehn twists given by the word w. Beginning with the first letter of w, label the meridian curves corresponding to an $a^{\pm}$ in w by α_i. The longitude of the torus will be denoted by β. Again beginning with the first letter, denote the boundary components of Σ by δ_i. We will use D_i to denote the positive Dehn twist about α_i, D_β for that about β and δ_i for the Dehn twists about

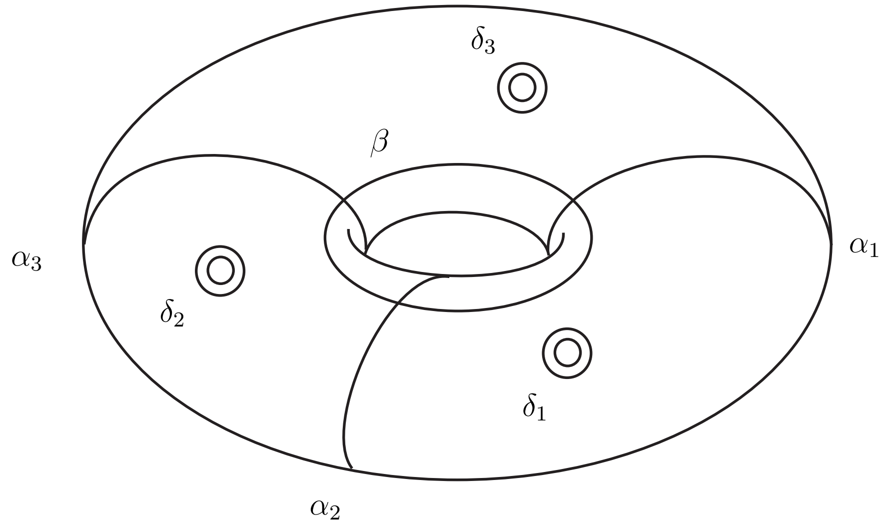


Figure 5.1: Curves on the 3-punctured torus used to present the star relation.

curves parallel to the boundary components δ_i. We will use δ to mean the total Dehn multitwist about all components of the boundary of the page.

Example. Using the open book decomposition of T^3 given by $(a^3b)^{-3}$ we label the three meridian curves α_i, $i = 1, 2, 3$. The longitude is β and the three boundary circles are δ_i, $i = 1, 2, 3$ as shown in Figure 5.1 for the star relation.

1. Elliptic: $|\mathrm{Tr}(A)| < 2$

   (a) $A = -S = a^{-1}b^{-1}a^{-1}$. In this case, T_A admits a Stein filling. Take the open book given by $a^{-1}b^{-1}a^{-1}$. The page is a punctured torus, and the monodromy is given by $\phi = \delta D_1^{-2}$. We use the chain relation on the punctured torus, $\delta = (D_1^2 D_\beta)^4$. Then $\phi = D_\beta(D_1^2 D_\beta)^3$, which gives a positive monodromy.

(b) $A = S = (a^{-1}b^{-1}a^{-1})^3$. T_A admits a Stein filling. Take the open book $(a^{-1}b^{-1}a^{-1})^3$ with $\phi = \delta D_1^{-2}D_2^{-2}D_2^{-2}$. Using the star relation $\delta = (D_1D_2D_3D_\beta)^3$, we can write $\phi = (D_1D_2D_3)^{-1}D_\beta(D_1D_2D_3)D_\beta D_1D_2D_3D_\beta$. Since this is a product of conjugates of positive Dehn twists, it is a positive monodromy.

(c) $A = -T^{-1}S = a^{-2}b^{-1}a^{-1}$. T_A admits a Stein filling. Take the open book $a^{-3}b^{-1}$ with $\phi = \delta D_1^{-3}$. Using the chain relation $\delta = (D_1^3D_2)^3$ we can write $\phi = D_\beta(D_1^3D_\beta)^2$.

(d) $A = -(T^{-1}S)^2 = a^{-1}b^{-1}$. T_A admits a Stein filling. Take the open book $a^{-1}b^{-1}$ with $\phi = \delta D_1^{-1}$. Use the chain relation get $\phi = D_\beta(D_1D_\beta)^5$.

(e) $A = T^{-1}S = (ba)^{-5} = a^{-2}b^{-1}a^{-3}b^{-1}a^{-2}b^{-1}$. T_A admits a Stein filling. Take the open book $a^{-2}b^{-1}a^{-3}b^{-1}a^{-2}b^{-1}$. Use the star relation to write

$$\phi = D_2(D_1D_2D_3)^{-2}D\beta(D_1D_2D_3)^2(D_1D_2D_3)^{-1}D\beta(D_1D_2D_3)D_\beta,$$

which is again a product of positive Dehn twists.

(f) $A = T^2{-}1S = (ba)^{-4}$. T_A admits a Stein filling. The open book corresponding to the word $(ba)^{-4}$ with monodromy $\phi = \delta D_1^{-1}D_2^{-1}D_4^{-1}D_4^{-1}$ can be written with a positive monodromy using a version of the star relation for the 4- (or more) punctured torus found by Korkmaz and Ozbagci in [24]. For ease, though, we apply the braid relation and choose the word $a^{-2}b^{-1}a^{-2}b^{-1}a^{-1}b^{-1}$. This gives $\phi = \delta D_1^{-2}D_2^{-2}D_3^{-1}$. Appealing again to the star relation (or adding positive twists to the previous example) yields a positive word representing ϕ.

2. Hyperbolic: $|\mathrm{Tr}(A)| > 2$

   (a) $A = b^{r'_0}ab^{r'_1}a\cdots b^{r'_k}a$ with $r'_i =\leq 0$ and $r'_0 < 0$. T_A admits a Stein filling. Indeed, the monodromy for the word $b^{r'_0}ab^{r'_1}a\cdots b^{r'_k}a$ is explicitly positive.

   (b) $A = (aba)^{-2}b^{r'_0}ab^{r'_1}a\cdots b^{r'_k}a$ with $r'_i = r_i + 2 \leq 0$ and $r'_0 < 0$. In this case, it is not clear when T_A admits a Stein filling. It is not likely that all these examples are Stein fillable (c.f. Example 2 of Section 4.7, the circle bundles over the Klein bottle with negative Euler number). However, if there are sufficiently many as in the word, we can repeatedly use the lantern relation to 'collect' the boundaries (in anywhere from 1 to 9 groups) and then apply the one of the chain or generalized star relations (of [24]) to realize the monodromy as positive. As an example, take the word $w = (aba)^{-2}b^{-2}a^3b^{-2}a^2b^{-2}a^5 \cong (aba)^{-2}a^2b^{-2}a^3b^{-2}a^2b^{-2}a^3$. Look at a segment $ab^{-2}a$. On this region, the page is a 4-punctured sphere and the monodromy is $\phi = \delta$. Applying the lantern relation, we get a new presentation $\phi = \alpha\beta\gamma$ (c.f. Section 4.4) where γ separates a pair of pants containing the two binding circles. Thinking of γ as a new boundary component of the page, we continue 'collecting' the boundaries into two groups, giving us a new presentation of the monodromy $\phi = \phi'\delta_1 D_1^{-2}\delta_2\delta'_3\delta'_4$, where ϕ' is positive. We can use the 4-star relation of [24] on the 4-punctured torus :$\delta = ((D_1D_3D_\beta)(D_2D_4D_\beta))^2$ to eliminate the D_1^{-2} and write ϕ as a positive word.

3. Parabolic: $|\mathrm{Tr}(A)| = 2$. $A = \pm T^n$, $n \in \mathbb{Z}$. Cases:

   (a) $A = T^n$, $n > 0$. We choose the word $A = (a^3b)^{-3}a^n$. As mentioned previously, these examples are given by Legendrian surgery on T^3 and so are Stein fillable.

   (b) $A = Id$. This is T^3. The factorization $A = (aba)^{-4} \cong (a^3b)^{-1}$ gives ξ_0, the unique Stein fillable contact structure on T^3, which was shown in the proof of Corollary 1 in Section 4.4. We recall the proof used only the star relation.

   (c) $A = T^n$, $n < 0$. Choose the factorization (of the conjugate) $A = b^n$, which is explicitly positive and so is Stein fillable.

   (d) $A = -T^n$, $n \in \mathbb{Z}$. The appropriate factorizations here are $A = (aba)^{-2}a^n$. When $n \geq 0$ we can use the chain relation on the twice punctured torus to realize ϕ as positive. When $n < 0$ though, the situation becomes much more interesting. For $n = -1, -2, -3$ one can (with a little effort) find a positive presentation for ϕ. When $n < -5$, though, Example 2 of Section 4.7 gives an argument why ϕ doesn't admit a positive presentation. This is very strong evidence that a contact structure compatible with w cannot be Stein fillable (though a proof does not yet exist). (Note: there is another likely candidate for the Stein fillable contact structure(s) on T_A, though construction of a compatible open book is not possible using the techniques of this paper.)

We'd also like to point out that the factorizations above can also apply to the minimally twisting but virtually overtwisted contact structures. The continued fraction blocks constructed in Section 4.6 have positive monodromies. This is gives us a mostly complete proof of the following theorem.

Theorem 5.0.5. *Let* $A = T^n$*,* $n < 0$ *or* $A = b^{r'_0}ab^{r'_1}a\cdots b^{r'_k}a$ *with* $r'_i =\leq 0$ *and* $r'_0 < 0$*. Then all of the tight, minimally twisting contact structures on* T_A *are Stein-fillable.*

Chapter 6

Universally Tight Contact Structures on S^1-bundles.

6.1 A Branched Cover Construction of $T^2 \times D^2$

A reader might notice that much of the detail above hinges on an understanding of the open book decomposition for (T^3, ξ_0). This was indeed the first example found, though the techniques were very different from the preceding material. Since it is of independent interest, we will discuss the construction here. Recall from the discussion from Chapter 5 that every Stein surface can be realized as a branched cover of B^4 along a positive, braided surface (which Rudolph has shown to be analytic). The boundaries of such surfaces are quasipositive braids and the branched cover on the 4-ball restricts to a branched cover on the boundary S^3. There is a Stein structure on $T^2 \times D^2$, given as a tubular neighborhood of the Clifford torus in $\mathbb{C}^2$. This has a natural Stein or symplectic product decomposition into two annuli factors, $\mathbb{A} \times \mathbb{A}$. Now, this has an easy description as a $\mathbb{Z}/2 \times \mathbb{Z}/2$ branched cover over $B^2 \times B^2$ given by branching each factor along a pair of points. (That is, branch $B^2 \times B^2$ along a pair of fibers $B^2 \times \{p_1, p_2\}$ to get $B^2 \times \mathbb{A}$, and then branch over a pair of annuli $\{q_1, q_2\} \times \mathbb{A}$ to get $\mathbb{A} \times \mathbb{A}$.) This is a 4-fold cover with immersed branch locus $\mathcal{B}$ given by the union of a pair of vertical fibers and a pair of horizontal fibers,

$$B^2 \times \{p_1, p_2\} \cup \{q_1, q_2\} \times B^2.$$

The boundary of the branch locus $\mathcal{B}$ is a 4-component link $\mathcal{L}$ given by taking the positive Hopf link and pushing off an unlinked copy as in Figure 6.1. This link lies nongenerically on the boundary trivial open book decomposition of S^3 (this has binding the unknot, thought of as the braid axis; the pages are disks); two components are transverse to the pages of the open book decomposition but two components lie on (distinct) pages. One can branch an open book decomposition along any link transverse to the fibers so we would like to perturb $\mathcal{L}$ to be transverse to the open book, i.e. a braid. There is an action by an arc of elements in $\mathrm{SU}(2)$ (beginning at the identity) that perturbs $\mathcal{B}$ so it is nongeneric with respect to the product decomposition on $\mathbb{C}^2$. This arc takes the components of $\mathcal{L}$ that lie on a page to the $(1, 1)$-cable of the transverse components. This puts the link into braid position and hence transverse to the open book and is shown in Figure 6.1. One convenience here is that the braid $\mathcal{L}$ is a pure braid (each strand connects back to itself) and so we can describe the braid in terms of Dehn twists in the mapping class group of the disk with 4 marked points, as in Figure 6.2. We then follow the branching description to determine what the page and monodromy of the cover are.

We branch over a single pair of marked points first. Each Dehn twist lifts to a pair of Dehn twists in the cover. Then monodromy on the annulus (now with 6 marked points) is given in Figure 6.3. The base locus in the cover is shown (in red). Taking the second branching, this time over 4 points, each Dehn twist again lifts to

Figure 6.1: The branch set $\mathcal{L}$ and its perturbation to a braid transverse to the open book.

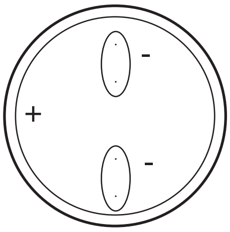


Figure 6.2: A Dehn twist presentation of the pure braid $\mathcal{L}$.

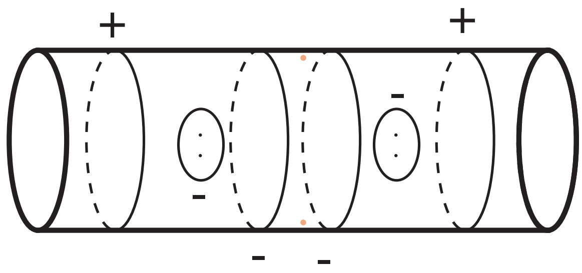


Figure 6.3: The monodromy and marked points on the annulus after the first 2-fold cover.

two in the cover. The page becomes a 4-punctured torus. The monodromy is shown in Figure 6.4. This open book decomposition corresponds to the word $(aba)^{-4}$, which can be shown to have a positive monodromy. Looking more closely, one can see that the Stein filling (that is, the PALF) corresponding to this positive word is actually $T^2 \times D^2$. Indeed, using the relations on the punctured torus given in [24], one can find positive presentations of the monodromies corresponding to all the words which are braid equivalent to $(aba)^{-4}$ *and do not contain the letter* a*, only* a^{-1}. Since each of these relations actually describes a Lefschetz pencil on $\mathbb{C}P^2$, it is a reasonable conjecture that all of the positive presentations for these nice open books of T^3 actually describe a Clifford torus in $\mathbb{C}P^2$ sitting nicely as a sub-fibration of the Lefschetz pencil.

This procedure can be generalized in many directions. The easiest is to increase the number of vertical or horizontal fibers in the base locus $\mathcal{B}$. This results in similar open book decompositions on the boundaries of the Stein surfaces $\Sigma \times \Sigma'$, where Σ and Σ' are any once- or twice-punctured surface.

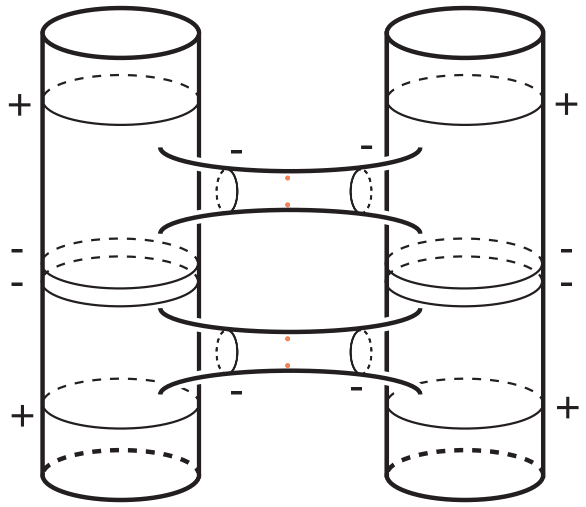


Figure 6.4: The final monodromy on the 4-punctured torus after the second 2-fold cover.

6.2 S^1-bundles

Once we've seen a few examples of open books created by branched covers, it becomes very easy to construct all of the S^1-invariant contact structures on circle bundles over any higher genus surface. We'll begin with a few examples and proceed to the general construction. The first example comes from the easy generalization of the construction of $T^2 \times D^2$. Branch over $2 \times 2n$ fibers to arrive at the diagram of $\mathfrak{ob}$ on $S^1 \times \Sigma$ in Figure 6.5, which can be isotoped to that of Figure 6.6. (Note: one can still follow the construction with an odd number of fibers, but the diagrams are less convenient, and since they aren't necessary we avoid presenting them.) Across each of the central necks, we see an $(aba)^{-2}$ diagram, which we know to be a $T^2 \times I$ segment where the pages and contact planes rotate by π to flip over. On the compliment of these regions, the monodromy is trivial and we use this

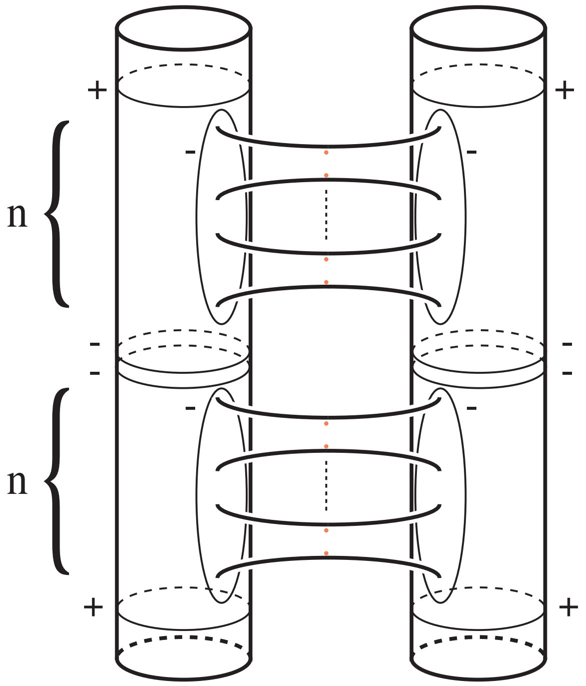


Figure 6.5: An open book decomposition of $S^1 \times \Sigma$ as the contact boundary of $\mathbb{A} \times F$.

description to build the open book in pieces. Let F_i be a twice punctured genus n surface, $i = 1, 2$. Take the trivial fibrations $F_i \times S^1$ and glue together along two $T^2 \times I$ segments I_1 and I_2 given by $(aba)^{-2}$. As oriented fibrations though, the S^1 direction of the total space and of the open book on $F_2 \times S^1$ (say) do not agree. However, we can still build a contact structure compatible with $\mathfrak{ob}$ that is invariant in the S^1 direction.

More generally, we can build open book decompositions compatible with all of the contact structures given by Honda (Theorem 2.11 part 3 of [22]) in his classification on circle bundles. The universally tight contact structures are determined

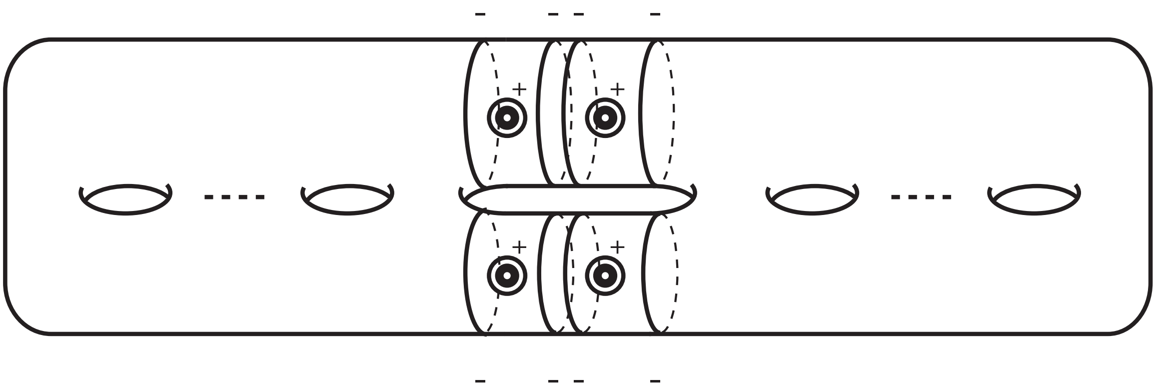


Figure 6.6: An easier visualization of the open book decomposition of $S^1 \times \Sigma$.

by a dividing set Γ on Σ and all are S^1-invariant. Let M be a circle bundle over a surface Σ. In the classification, one defines a projection function $\pi : (M, \xi) \to D$ that takes ξ to the dividing set of a minimal 'pseudo-section' of M. One cuts M along a convex torus T tangent to the S^1 fibers and with vertical dividing curves and looks at the dividing set of a minimal convex section of $M' = M \backslash T = S^1 \times \Sigma'$, where Σ' is Σ cut along a non-separating curve. To build and open book decomposition compatible with such an example, we begin with any possible dividing set on Σ (without homotopically trivial components), build the compatible open book for $S^1 \times \Sigma$ and describe how it needs to be altered to arise at an open book for a non-trivial circle bundle.

Theorem 6.2.1. *Let Γ be a dividing set on Σ without homotopically trivial components. Let ξ be the associated universally tight contact structure on $S^1 \times \Sigma$ (with $t_{S^1} = 0$). Then an open book $\mathfrak{ob}$ carrying ξ can be described as follows. Begin with Σ as the page. For every component $\gamma \subset \Gamma$, replace an interval neighborhood of*

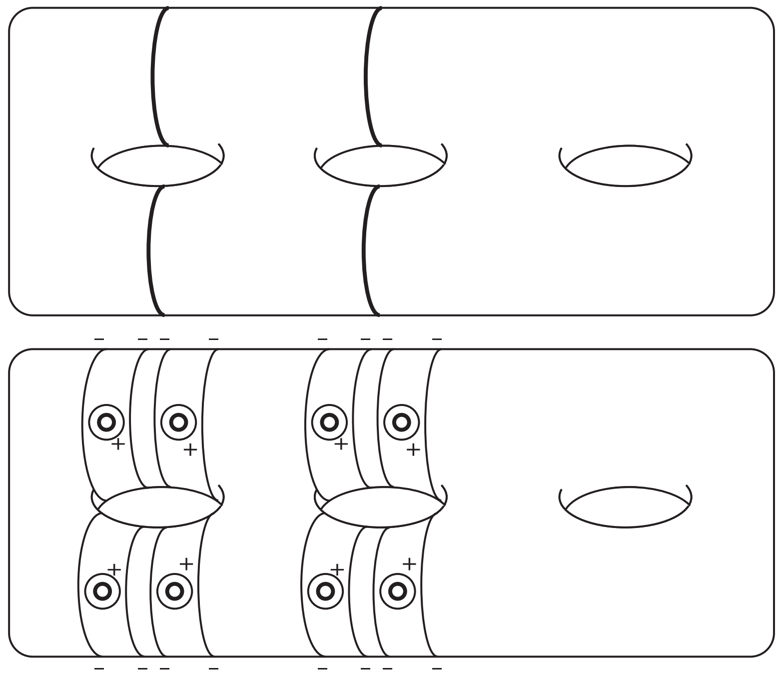


Figure 6.7: A dividing set on Σ and its corresponding open book decomposition.

$\gamma \subset \Sigma$ *with the open book segment* $(aba)^{-2}$.

An example is given in Figure 6.7.

Proof. Notice first that if we swap the positive and negative regions, the contact structures are isomorphic (simultaneously change the orientations on S^1 and Σ) and so the abstract open book doesn't notice which region is which. The construction begins as above, first by finding an embedding into $S^1 \times \Sigma$. We then show there is a compatible S^1-invariant contact structure that realizes $p \times \Sigma$ as convex with dividing set Γ. The first part of this we discussed already. Denote the components of $\Sigma \backslash \Gamma$ by F_i, $i = 1, \dots m$ and the components of Γ by γ_j, $j = 1, \dots n$. Build the open book by taking the trivial fibrations $F_i \times S^1$ attaching by $T^2 \times I$ segments with relative open book decompositions $(aba)^{-2}$. We build the compatible contact structure the same way, making sure each piece is S^1-invariant and ensuring the S^1 factors glue together nicely. On each F_i, construct a contact structure following the model in the Thurston-Winkelnkemper construction. Since the monodromy is trivial, the contact structure is S^1-invariant and the boundary is a union of prelagrangian tori where the foliation on each by the contact planes can be made arbitrarily close to the pages. That is to say, the contact planes begin rotating just past the angle of the pages. On each I_j segment, we take the contact form given by $\alpha = \sin(t)dx + \cos(t)dy$ in coordinates (x, y, t) on $T^2, \times[0, \pi]$. The fibers of $S^1 \times \Sigma$ intersect I_j along the annuli $y = const.$ and agree with the pages near the boundary of I_j and everything is invariant in the y direction. In order to glue, we extend our interval to $[-\epsilon, \pi + \epsilon]$, and then rescale α in the t direction near the boundaries, ensuring the contact planes glue smoothly to those on the F_i. This doesn't change the y-invariance and since the gluing matches up the S^1 and y directions (sometimes switching orientations), the resulting contact structure is S^1-invariant and hence universally tight. Further, using the S^1 factor as a contact vector field, any fiber $p \times \Sigma$ is convex with dividing set Γ (further this dividing set is minimal among all convex surfaces isotopic to a fiber). We may also use this to guarantee compatibility (or Proposition 3.0.6, since we could just have easily have matched the S^1 direction to the Reeb field for α, giving a contact vector field on M which is positively transverse to every page of the open book and to the contact planes, ensuring the existence of a contact form that restricts to a primitive form on each of the pages.

We may modify this picture slightly and construct open books for the non-trivial circle bundles. When the Euler class is negative, we can achieve this by sprinkling b^{-1} segments around $S^1 \times F_i$ portions of the open book for $S^1 \times \Sigma$. Further, the contact structure is independent of how these are distributed as the open books are all stably equivalent. To see this, isotope the diagram so a puncture is adjacent to an I_j, giving an $(aba)^{-2}b^{-1}$ region, which is braid equivalent to $b^{-1}(aba)^{-2}$.

For $e(M) > 0$, we perform Legendrian surgery on the contact structure on $S^1 \times \Sigma$ corresponding to the same dividing set Γ. Along each I_j there is a T^2 of vertical Legendrian curves (lying on the page of $(aba)^{-2}$) with twisting number 0. Legendrian surgery inserts an a into the middle of this word and is smooth (-1)-surgery, giving a circle bundle with positive Euler number. Again, one can show the contact structure is independent of the choices by showing the open books are stably equivalent. To push an a from one I segment to another, connect the two regions with a one handle and canceling Dehn twist along a curve c that intersects each 'middle' region exactly once. Using a sequence of braid relations, we can turn one a on I into another c and then back into an a on a different I and then destabilize. $\square$

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Vita

Jeremy Van Horn-Morris has a beautiful wife and they have two adorable children, both born in Texas. He began graduate studies at the University of Texas at Austin in 2001, after completing his undergraduate studies at the University of Oregon. There in 1996 he began work in Chemistry with poor lab skills before eventually finding his way to mathematics, which he cannot seem to leave. He was born in Springfield, Oregon on October 27th, 1978. At 9 lbs. he was the runt and the first of six to Kim Van Horn and John Morris. He is still the shortest. On occasion, they are indistinguishable. One is greatly missed.

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This dissertation was typeset with LaTeX[†] by the author.

[†] LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.