L-functions from an analytic perspective

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$Z(t)$ in two regions
Facts:

- \[ N(T) = \frac{1}{2\pi} T \log \left( \frac{T}{2\pi e} \right) + O(\log t), \]
  where \( N(T) \) = number of zeros in \([0, T]\).
  So the average gap \( \sim 2\pi/\log T \),

- \[ \frac{1}{T} \int_0^T Z(t) \, dt \to 0 \text{ as } T \to 0. \]

Conjectures:

- \( \text{Prob}(\gamma_{j+1} - \gamma_j < \frac{\varepsilon}{\log j}) \approx \varepsilon^3 \), where the zeros are \( \gamma_1 < \gamma_2 < \gamma_3 < \cdots \).

- \( Z(t) \) gets as large as \( \exp\left(\sqrt{\frac{1}{2} \log T \log \log T}\right) \), and no larger.

- Local maxima of \( Z(t) \) are positive, and local minima are negative.
The Hardy $Z$-function

The Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \sigma > 1 \quad (1)$$

Meromorphic continuation to all $s \in \mathbb{C}$. Functional equation:

$$\Lambda(s) := \pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \Lambda(1 - s)$$

Equivalently,

$$\zeta(s) = \chi(s) \zeta(1 - s)$$

Equivalently,

$$Z(t) := \chi\left(\frac{1}{2} + it\right)^{-1/2} \zeta\left(\frac{1}{2} + it\right)$$

is real if $t$ is real.
The spacing between zeros

$N(T)$: the number of zeros in $0 < t < T$.

$$N(T) = \frac{1}{2\pi} T \log \left( \frac{T}{2\pi e} \right) + S(t)$$

$$S(T) = O(\log T)$$

Average gap between zeros: solve $N(T + \delta) - N(T) \approx 1$.

$$\delta N'(T) \approx 1$$

$$\delta \approx \frac{1}{N'(T)} = \frac{2\pi}{\log(T/2\pi)}$$
Mathematica demo about rigidity in the zero spacing.
Fourier transform of the $Z$-function
Unsolved problems on zero spacing

Let $\tilde{\gamma} = \frac{1}{2\pi} \gamma \log(\gamma)$, so that $\tilde{\gamma}_{j+1} - \tilde{\gamma}_j$ is 1 on average.

- Show that $\tilde{\gamma}_{j+1} - \tilde{\gamma}_j < 0.49$ a positive proportion of the time. (Related to $L(1, \chi)$; Thursday lecture.) Current record: 0.51???

- Show that for all $A > 1$, there exists $j$ so that

$$\tilde{\gamma}_{j+1} - \tilde{\gamma}_j > A$$

Current record: maybe 3?

Some methods involve comparing

$$\int_0^T M(t)^2 Z(t)^2 dt \quad \text{to} \quad \sum_{\gamma < T} \int_{-c/\log T}^{c/\log T} M(\gamma + t)^2 Z(\gamma + t)^2 dt$$