DOTS, BOXES, PILES and HOLES

MIDDLE-SCHOOL MATERIALS

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This material represents sample chapters from the MIDDLE-SCHOOL series:

MATHEMATICAL THINKING!

available at the above website. These are draft materials for an on-line experience to be developed.

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SERIES CONTENT

EXPLORATION 1: THE POWER OF A PICTURE (22 pages)
How long do you think it would take to add up all the numbers from one to a million and back down again? In this exploration you will learn how to do it in under one second!
TOPICS COVERED: The counting numbers; their addition and multiplication. Clever sums and figurate numbers.

EXPLORATION 2: FACTORS AND PRIMES (12 pages)
Open doors and closed doors. Learn how locker doors reveal patterns in factors of numbers.
TOPICS COVERED: Factors of numbers. Composite numbers and prime numbers. The great locker experiment!

EXPLORATION 3: DOTS AND BOXES (26 pages)
Exploding dots reveal the secret to understanding how numbers are written and added and multiplied and …
TOPICS COVERED: Place-value and its role in arithmetic algorithms.

EXPLORATION 4: DIVISION (22 pages)
The mysterious long-division algorithm is finally explained. It makes sense!
TOPICS COVERED: Long division via dots and boxes. Extension to polynomial division.

EXPLORATION 5: PILES AND HOLES (24 pages)
How a story that isn't true can explain the mysteries of negative numbers!
TOPICS COVERED: Basic introduction to negative numbers: addition and "subtraction," and more polynomials.

EXPLORATION 6: THE VINCULUM AND PARENTHESES (14 pages)
A snazzy symbol from the 1500s makes order of operations easy and obvious. Why don't we use it today?
TOPICS COVERED: Parentheses via the vinculum. Basic order of operations via the vinculum.

EXPLORATION 7: TROUBLESOME ZERO (16 pages)
Is zero easy to work with or tricky to work with? Is it even a number? Let's play with tricky zero and sort out all its sneaky behaviors.
TOPICS COVERED: Is zero a number? Basic arithmetic with zero, and the danger of division. The use of zero in base 10 arithmetic.

EXPLORATION 8: MULTIPLICATION (26 pages)
Slicing rectangles and slicing cheese is all one needs to multiply lengthy numbers (even if those rectangles and cheeses are negatively long and wide – and high!)
TOPICS COVERED: Expanding brackets and long-multiplication. Why the product of two negative numbers is positive.

EXPLORATION 9: ACTING CLEVERLY (14 pages)
Clever ways to avoid ghastly work! And clever ways to amaze your friends!
TOPICS COVERED: Grouping and factoring. Estimation. Mathematical party tricks.

EXPLORATION 10: FRACTIONS: ADDING AND SUBTRACTING (24 pages)
Why do people find this topic so scary? Pies and boys (and pies and girls) makes it easy!
TOPICS COVERED: Fractions and their basic arithmetic.

EXPLORATION 11: FRACTIONS: MULTIPLYING AND DIVIDING (32 pages)
Pies and Boys continue to save the day as we explore further properties of fractions and all the annoying jargon that goes with them.
TOPICS COVERED: More arithmetic of fractions. Mixed numbers. Extension to Egyptian fractions.

EXPLORATION 12: EXPONENTS (18 pages)
Does folding a piece of paper in half multiple times explain everything there is to know about the powers of two?
TOPICS COVERED: Exponents and their properties

EXPLORATION 13: RETURN OF THE VINCULUM (28 pages)
The vinculum returns to save the day in problems with division!
TOPICS COVERED: The use of vinculum in fractions. Division in algebra. Division as multiplication by fractions.

EXPLORATION 14: DECIMALS (34 pages)
Exploding and unexploding dots return to explain decimals!
TOPICS COVERED: Decimals and their arithmetic.

EXPLORATION 15: EQUALITY AND INEQUALITY (26 pages)
The mathematics of staying balanced – or staying unbalanced if you prefer!
TOPICS COVERED: Properties of equality. Properties of inequality. Ranges of values.

EXPLORATION 16: CONFUSING THINGS THAT ARE THE SAME (22 pages)
Let's be honest: Many things in mathematics are confusing. Let's see if we can sort out a slew of confusing things once and for all.
TOPICS COVERED: The word "more" and the word "less." Percentages. The word "of." Ratios. The equals sign.

EXPLORATION 17: NUMBERS ON THE LINE (22 pages)
How much of the number line actually has anything to do with numbers?
TOPICS COVERED: Development of the number line. Rational and irrational numbers and the hierarchy of numbers. Repeating decimals. Attempts to define the reals.

EXPLORATION 18: SOLUTIONS

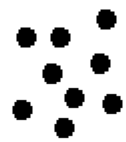
MATHEMATICAL THINKING!

Exploration 3

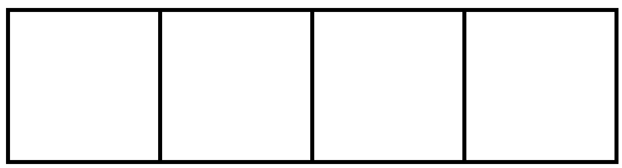
DOTS AND BOXES

A. GETTING STARTED

Here are some dots (nine of them, I believe):



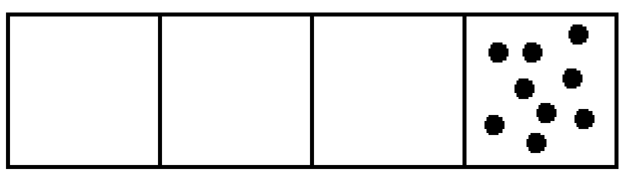
and here are some boxes:



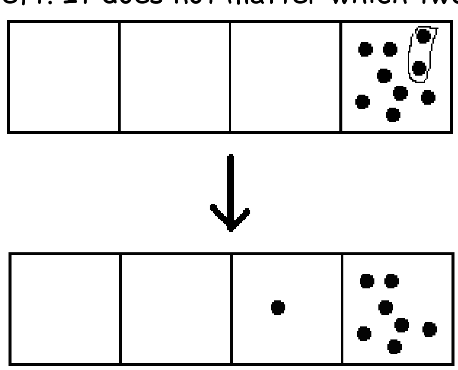
In the game we are about to play boxes explode dots! In fact, these boxes like to follow the following rule:

> **THE $1 \leftarrow 2$ RULE:**
> **Whenever there are two dots in any one box they "explode," disappear and become one dot in the next box to their left**

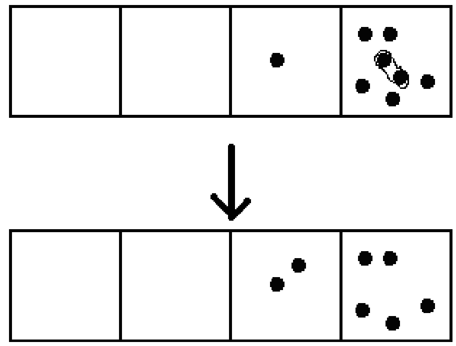
We start by placing our nine dots in the right-most box:



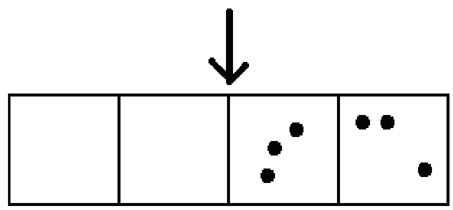
There are certainly two dots somewhere in this box and they explode to become one dot one place to the left. It does not matter which two dots we circle.



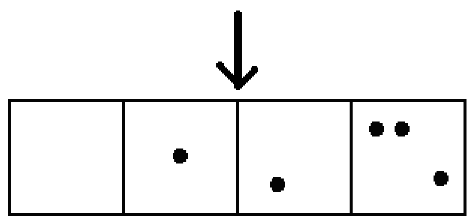
And it can happen again:



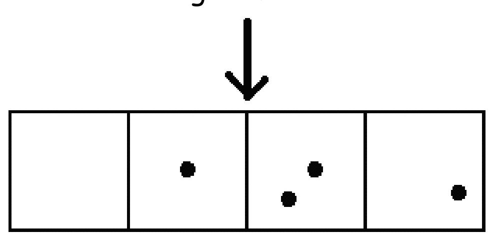
And again!



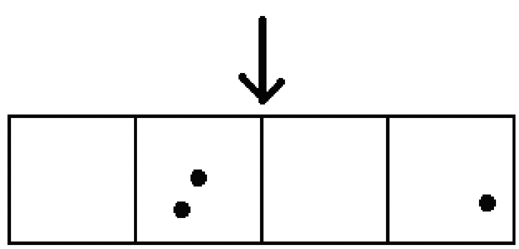
And it can happen again in right-most box, but it can now also happen in the second box. Let's do it here now just for fun:



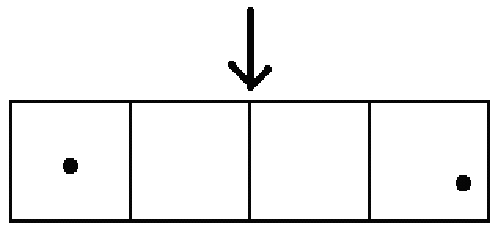
Okay, now we have to go back to the right-most box:



And another time:



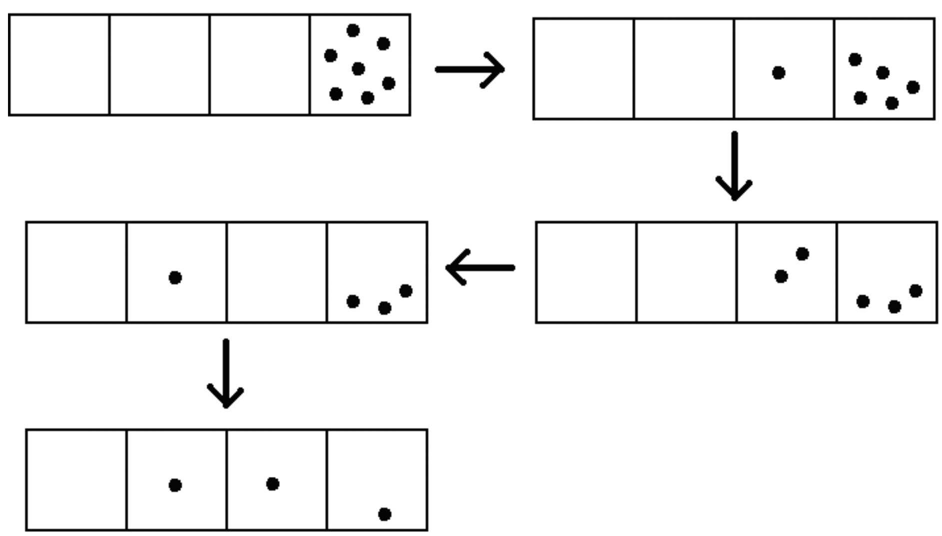
And one final time!



After all this, reading from left to right we are left with one dot, followed by zero dots, zero dots, and one final dot. Let's say:

OUR CODE FOR THE NUMBER 9 IS: 1001

Here's what happens with seven dots:

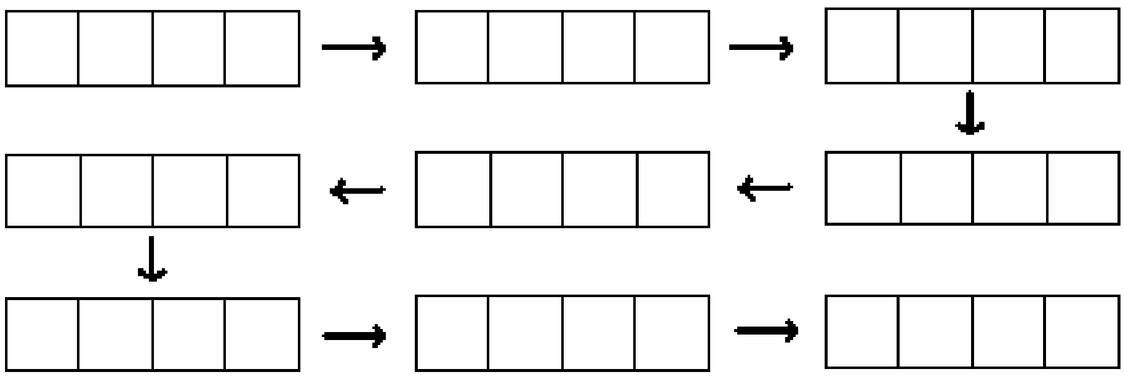


EXERCISE: Circle the pair of dots that "exploded" at each turn in the above diagram.

OUR CODE FOR THE NUMBER 7 IS: 0111

Your turn!

Question 1: Draw 10 dots in the right-most box and perform the explosions. What is our code for the number ten?



OUR CODE FOR TEN IS: _______________

Question 2: Drawing this on paper is hard. Maybe you could use buttons or pennies for dots and do this by hand. What could you use for the boxes?

Use your chosen objects to find the code for the number 13.
Also find the code for the number 6.

OUR CODE FOR 13 IS: _______________

OUR CODE FOR 6 IS: _______________

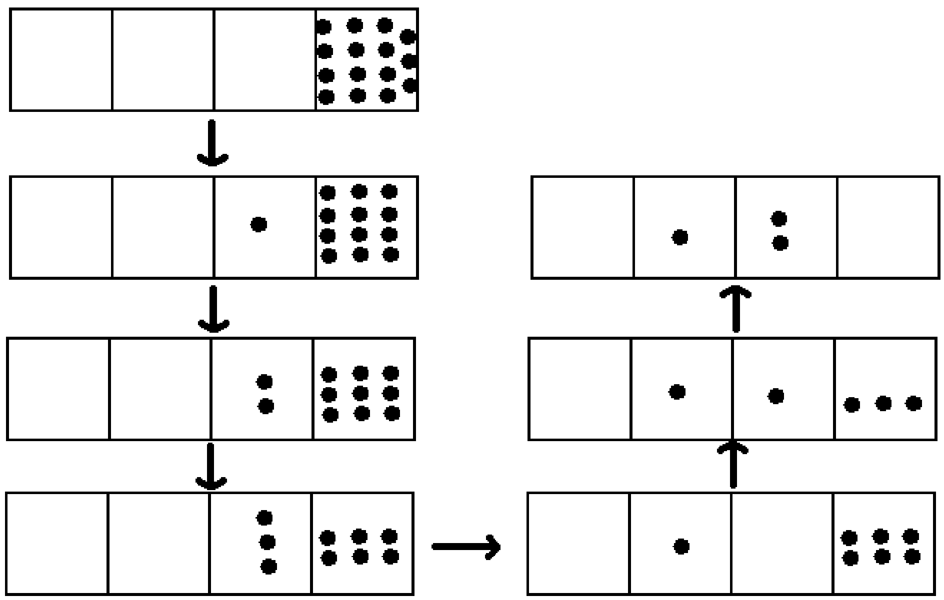
Question 3: CHALLENGE: What number has code 0101?

B. OTHER RULES

Let's play the dots and boxes game but this time with …

> **THE $1 \leftarrow 3$ RULE:**
> **Whenever there are three dots in any one box they "explode," disappear and become one dot in the next box to their left**

Here's what happens to fifteen dots:



We have:

THE $1 \leftarrow 3$ CODE FOR FIFTEEN IS: 0120

Question 4: a) Show that the $1 \leftarrow 3$ code for twenty is 0202.
b) Show that the $1 \leftarrow 3$ code for four is 0011.

Question 5: What is the $1 \leftarrow 3$ code for 13? For 25?

Question 6: Is it possible for a number to have $1 \leftarrow 3$ code 2031? Explain.

Question 7: HARD CHALLENGE: What number has $1 \leftarrow 3$ code 1022?

Let's keep going …

Question 8: What do you think is the $1 \leftarrow 4$ rule?
What is the $1 \leftarrow 4$ code for the number thirteen?

Question 9: What is the $1 \leftarrow 5$ code for the number thirteen?

Question 10: What is the $1 \leftarrow 9$ code for the number thirteen?

Question 11: What is the $1 \leftarrow 5$ code for the number twelve?

Question 12: What is the $1 \leftarrow 9$ code for the number thirty?

AHA MOMENT!

Question 13: What is the $1 \leftarrow 10$ code for the number thirteen?

What is the $1 \leftarrow 10$ code for the number thirty-seven?

What is the $1 \leftarrow 10$ code for the number 238?

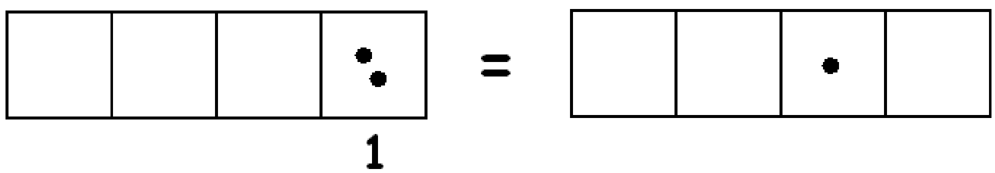
What is the $1 \leftarrow 10$ code for the number 5834?

C. WHAT'S REALLY GOING ON

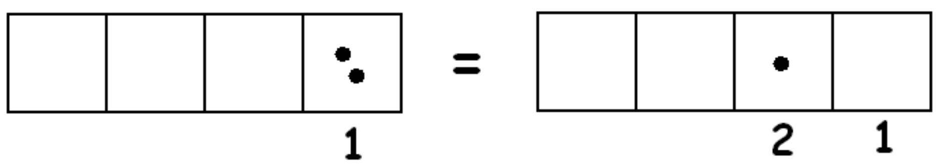
Let's go back to the $1 \leftarrow 2$ rule for a moment.

> **THE $1 \leftarrow 2$ RULE:**
> **Whenever there are two dots in any one box they "explode," disappear and become one dot in the next box to their left**

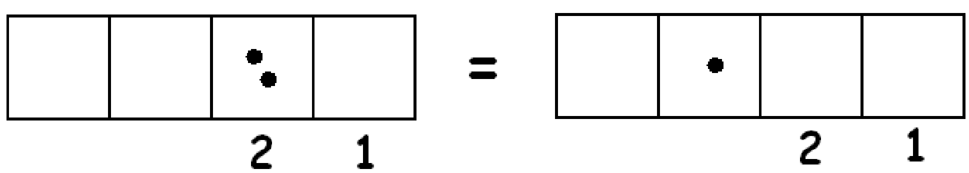
Two dots in the right-most box is worth one dot in the next box to the left.



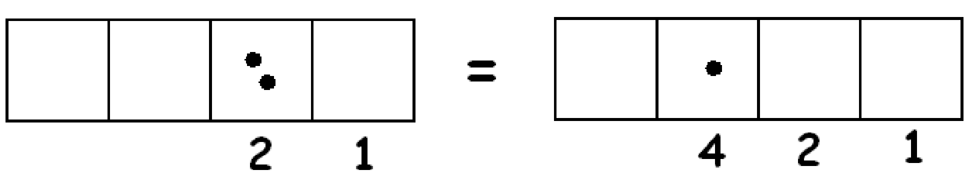
If each of the original dots is worth "one," then the single dot on the left must be worth two.



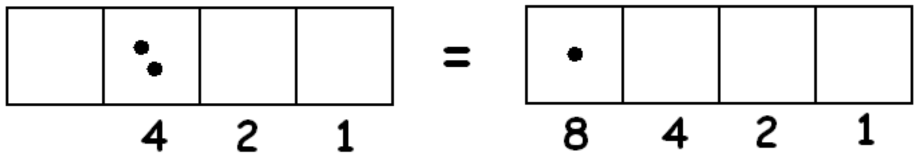
But we also have two dots in the box of value 2 is worth 1 dot in the box just to the left …



This next box must be worth two 2s. That's four!

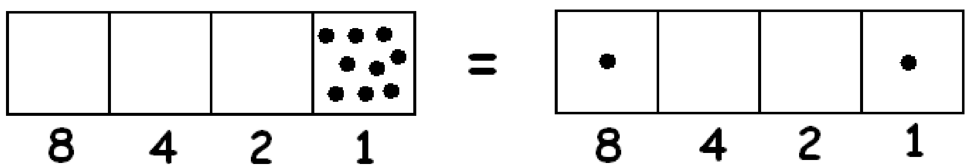


And two of these fours makes 8.



Question 14: If there was one more box to the left, what must it be worth?

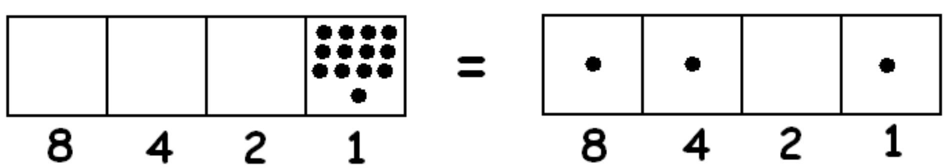
We said earlier that the $1 \leftarrow 2$ code for 9 was 1001. Let's check:



Yep! Nine 1s does equal one 8 plus one 1.

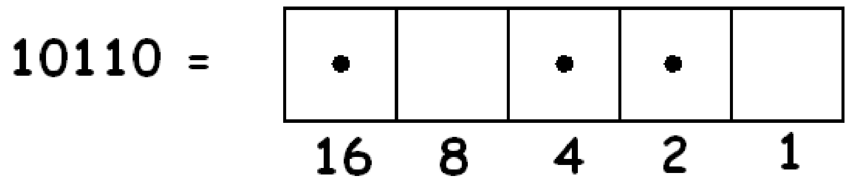
$$9 = 8 + 1$$

We also said that 13 has code 1101. This is correct.



$$13 = 8 + 4 + 1$$

What number has code 10110? Easy:



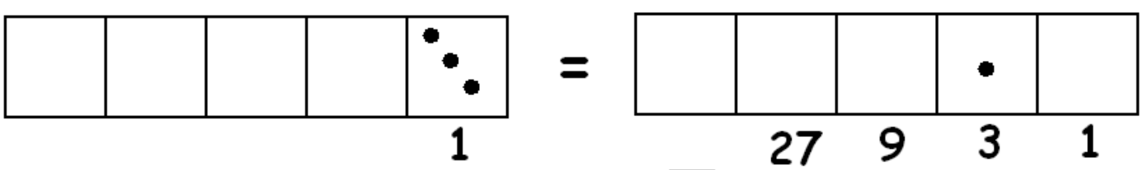
$$16+4+2=22$$

Question 15: What number has $1 \leftarrow 2$ code 100101 ?

Question 16: What is the $1 \leftarrow 2$ code for the number two hundred?

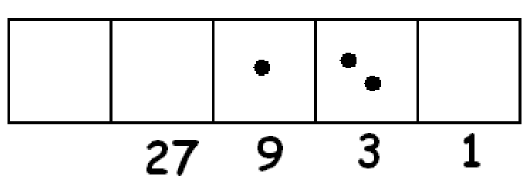
FANCY LANGUAGE: People call numbers written in $1 \leftarrow 2$ code binary numbers. (The prefix *bi* means "two.")

Question 17: In the $1 \leftarrow 3$ system we have three dots in one box is worth one dot one place to the left. This gives the numbers …



a) What is the next number?

We said that the $1 \leftarrow 3$ code for fifteen is 120. We see that this is correct because …



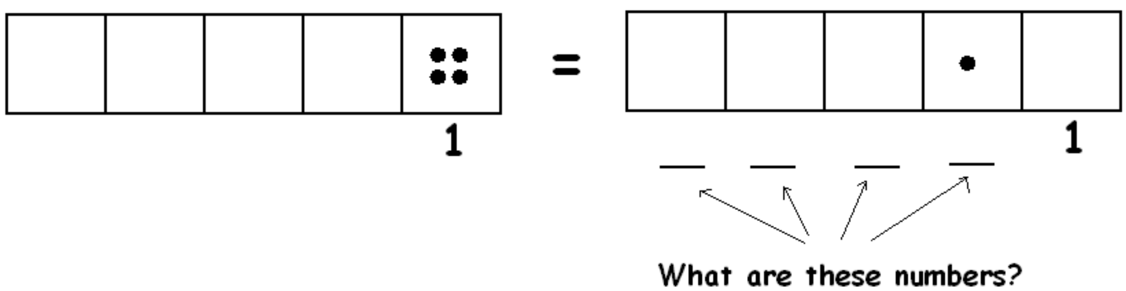
$$9+3+3=15$$

b) Actually the $1 \leftarrow 3$ code for fifteen was 0120. Is it okay to leave off the first zero? Is it okay to leave off the last zero instead and write 012 ?

c) What number has $1 \leftarrow 3$ code 21002 ?

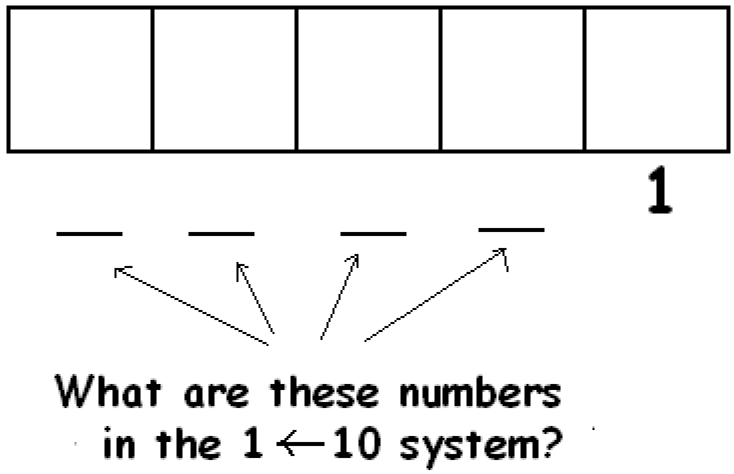
d) What is the $1 \leftarrow 3$ code for two hundred?

Question 18: In the $1 \leftarrow 4$ system four dots in one box is worth one dot one place to the left. What is the value of each box?



a) What is the $1 \leftarrow 4$ code for 29?
b) What number has $1 \leftarrow 4$ code 132?

Question 19: a)



b) What is the code for the number 98723 in the $1 \leftarrow 10$ system?

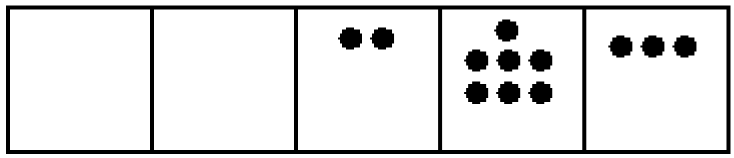
c) When we write the number 7842 the "7" is represents what quantity? The "4" is four groups of what value? The "8" is eight groups of what value? The 2 is two groups of what value?

d) Why do human beings like the $1 \leftarrow 10$ system for writing numbers? Why the number 10? What do we like to use on the human body for counting?

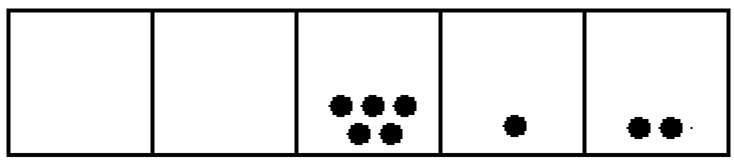
Would Martians likely use the $1 \leftarrow 10$ system for their mathematics? Why or why not?

D. USING THE $1 \leftarrow 10$ SYSTEM

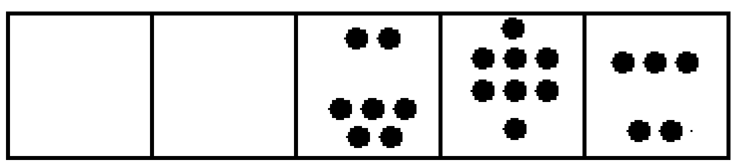
Here is the number 273 in the $1 \leftarrow 10$ system:

273 = 

And here is the number 512:

512 = 

If we add these, we obtain:



This is the number 785. We've just worked out the sum:

$$\begin{array}{r} 273 \\ +\ 512 \\ \hline =\ 785 \end{array}$$

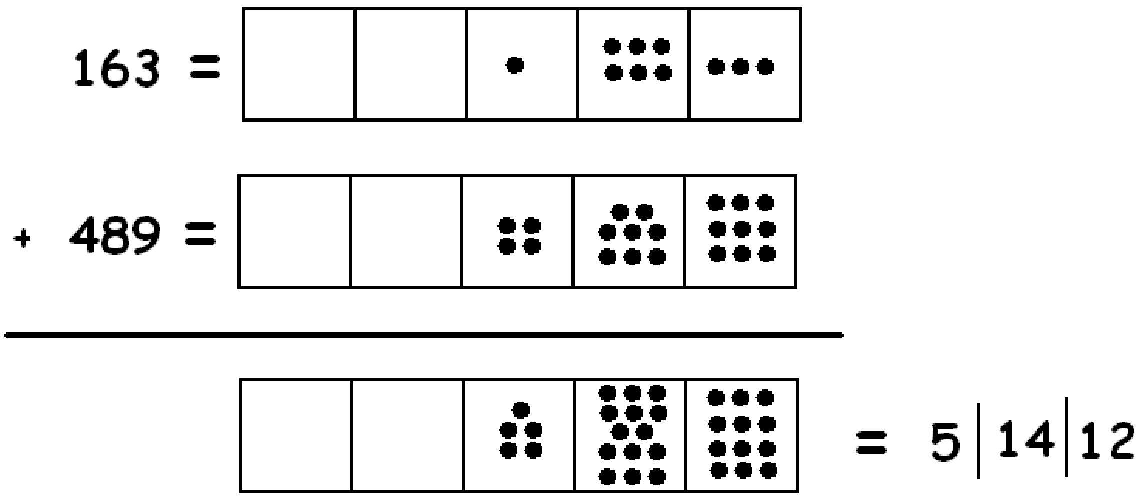
And saying out the long way we have:

> Two hundreds plus five hundreds gives 7 hundreds
> Seven ones plus one ten gives 8 tens
> Three ones plus 2 units gives 5 units

This is the answer 785.

Question 20: Draw the dots and boxes $1 \leftarrow 10$ picture for the number 3704. Add to this the picture for 2214. What is $3704 + 2214$?

Let's do another one. Consider $163 + 489$.



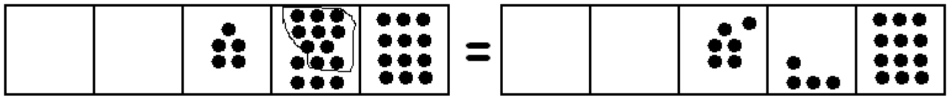
$$\begin{array}{r} 163 \\ +\,489 \\ \hline 5\ \ 14\ \ 12 \end{array}$$

And this is absolutely mathematically correct:

One hundred plus four hundreds does give 5 hundreds
Six tens plus eight tens does give 14 tens
Three ones plus nine ones does give 12 ones.

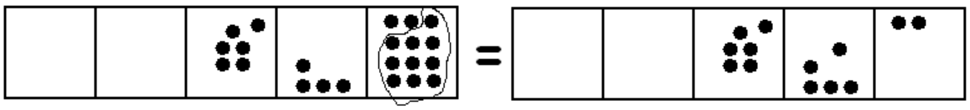
The answer is 5 | 14 | 12 which we might try to pronounce as "five hundred and fourteeny-tenty twelvety"! (Oh my!)

The trouble with this answer – though correct - is that most of the rest of the world wouldn't understand what we are talking about! Since this is a $1 \leftarrow 10$ system we can do some explosions.



$= 6|4|12$

The answer is "six hundred forty twelvety"! Still correct, but let's do another explosion:



$= 6|5|2$

The answer is "six hundred fifty two." Okay, the world can understand this one!

$$\begin{array}{r} 163 \\ +\,489 \\ \hline 5\,|\,14\,|\,12 \end{array} = 652$$

Question 21: Solve the following problems thinking about the dots and boxes. (You don't have to draw the pictures; just do it!) And then translate the answer into something the rest of the world can understand.

$$\begin{array}{r} 148 \\ +\,323 \\ \hline = \quad \end{array} \qquad \begin{array}{r} 567 \\ +\,271 \\ \hline = \quad \end{array} \qquad \begin{array}{r} 377 \\ +\,188 \\ \hline = \quad \end{array} \qquad \begin{array}{r} 582 \\ +\,714 \\ \hline = \quad \end{array}$$

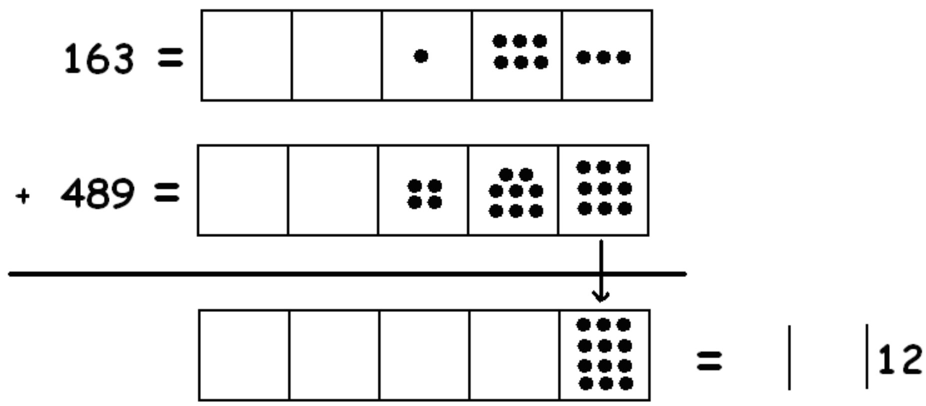
$$\begin{array}{r} 310462872 \\ +\,389107123 \\ \hline = \quad \end{array} \qquad \begin{array}{r} 87263716381 \\ +\,18778274824 \\ \hline = \quad \end{array}$$

WHAT TEACHER'S TEACH …

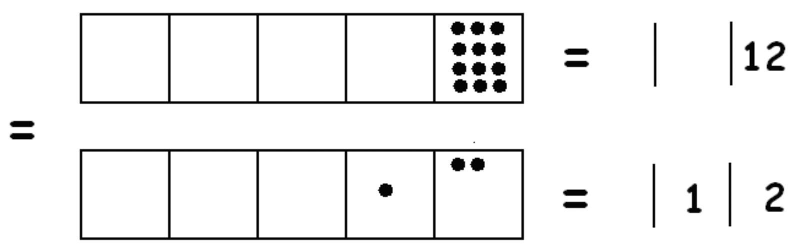
Let's go back to the example $163+489$. Some teachers don't like writing:

$$\begin{array}{r} 163 \\ +\,\underline{489} \\ 5\,|\,14\,|\,12 \end{array} = 652$$

They prefer to teach their students to start with the 3 and 9 at the end and sum those to get 12. This is of course correct – we got 12 as well.



But they don't want students to write or think "twelvety" and so they have them explode ten dots:

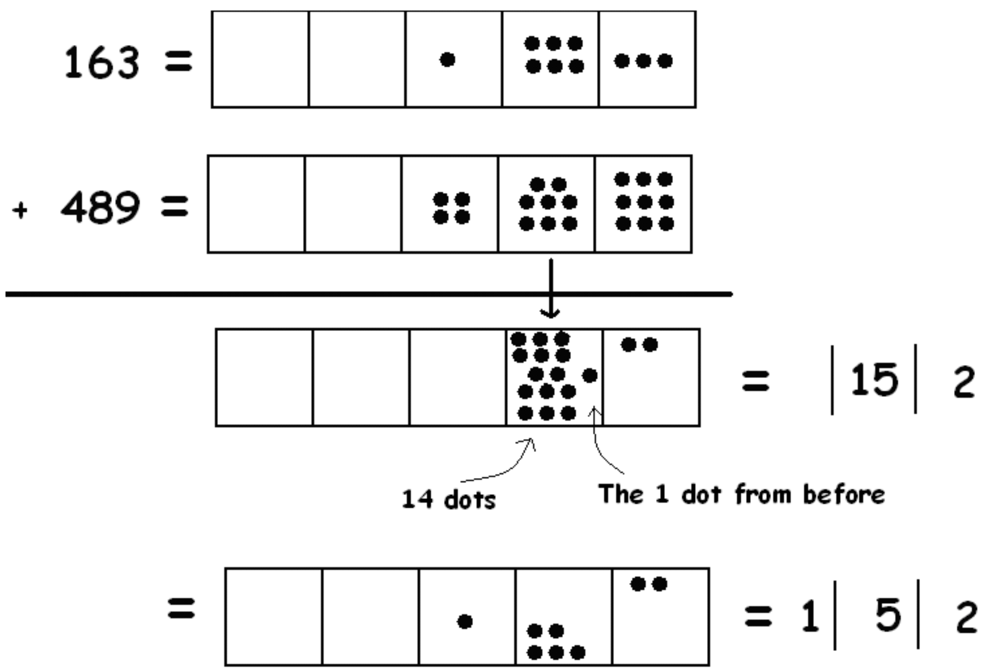


and they teach their students to write:

$$\begin{array}{r} 1 \\ 163 \\ +\,\underline{489} \\ 2 \end{array}$$

which seems mysterious. But it makes sense to us now because have put that "1" in the tens place which is exactly what an explosion does.

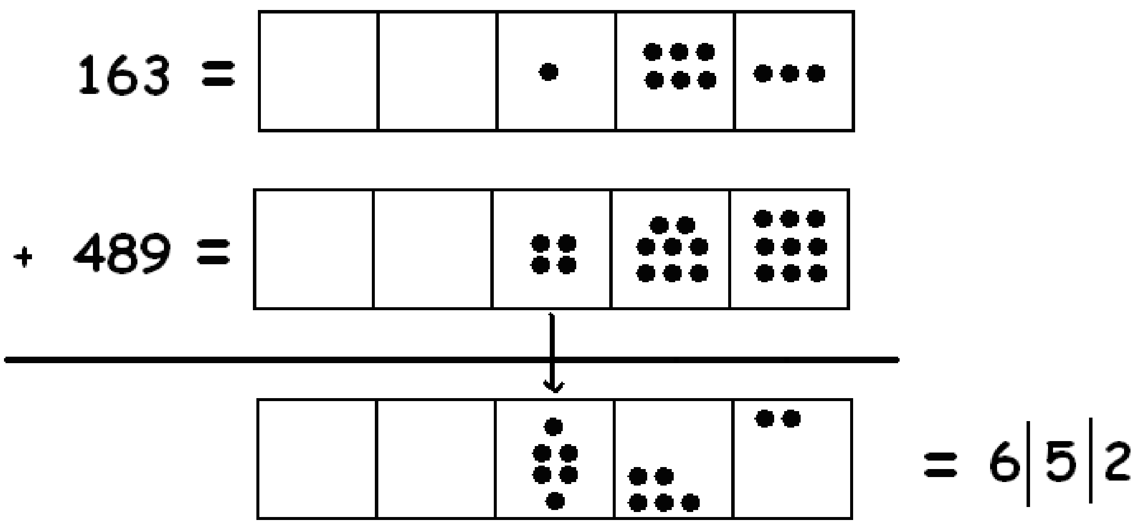
Now we carry on with the problem and add the tens:



and students are taught to write:

$$\begin{array}{r} 1 \\ 163 \\ +\,\underline{489} \\ 52 \end{array}$$

And now we finish the problem.



and we write:

$$\begin{array}{r} {}^{1} \\ 163 \\ +\ \underline{489} \\ 652 \end{array}$$

Teachers like to teach their students to do all their explosions first. This means that students have to start at the right of the problem and work towards the left "carrying" digits that come from the explosions.

In the dots and boxes method one adds in any direction or order one likes and does the explosions at the end.

WHY DO TEACHERS LIKE THEIR METHOD? Because it is efficient.

WHY DO WE LIKE THE DOTS AND BOXES METHOD? Because it is easy to understand.

Question 22: Redo the problems of question 2 the teachers' way. You will see that it is quicker.

Question 23: HARD CHALLENGE. Here is an addition problem in a $1 \leftarrow 5$ system. (THIS IS NOT $1 \leftarrow 10$).

$$\begin{array}{r} 20413 \\ +\ 13244 \\ \hline \end{array}$$

a) What is the $1 \leftarrow 5$ answer?
b) If this were an addition problem in a $1 \leftarrow 10$ system, what would the answer be?

Question 24: Jenny was asked to compute 243192×4. She wrote:

$$243192 \times 4 = 8|16|12|4|36|8$$

a) What was she thinking? Why is this a mathematically correct answer?
b) Translate the answer into a number that the rest of the world can understand.

c) Find the answers to these multiplication problems:

$156 \times 3 =$

$2873 \times 2 =$

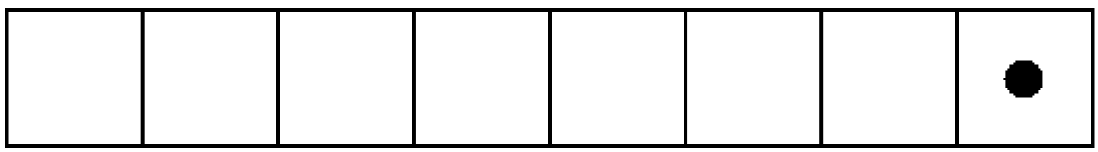
$71181 \times 5 =$

$3726510392 \times 2 =$

$765 \times 9 =$

E. EXPLORATION AND ADVENTURE

ADVENTURE 1: Jay decides to play with a system that follows a $1 \leftarrow 1$ rule. He puts one dot into the right-most box. What happens? (Suppose the row of boxes is infinitely long to the left!)



Suggi plays with a system following the rule $2 \leftarrow 1$. She puts one dot into the right-most box. What happens for her?

ADVENTURE 2: Poindexter decides to play with a system that follows the rule $2 \leftarrow 3$.

a) Describe what this rule does when there are three dots in a box.

b) Draw diagrams or use buttons or pennies to find the $2 \leftarrow 3$ codes for the following numbers:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 24, 27, 30, 33, 36, and 39

Any patterns?

c) Repeat this exercise for your own rule. Choose two numbers *a* and *b* and figure out what the code is for your $a \leftarrow b$ system for each of the numbers above.

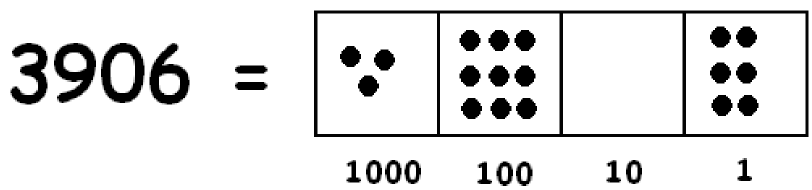
MATHEMATICAL THINKING!
Exploration 4

DIVISION

A. GETTING STARTED

Let's keep playing with the $1 \leftarrow 10$ system of dots and boxes. (After all, it represents the way human beings like to write their numbers!)

Here is the number 3906 in the $1 \leftarrow 10$ system:



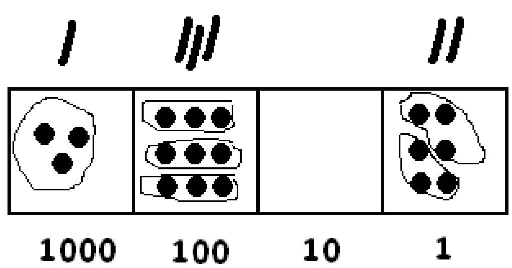
Let's divide this number by three. This means that we are asking:

How many groups of 3 fit into 3906?

Now, three dots looks like ●●● so we are really asking:

How many groups of ●●● can we see in the picture?

There is certainly one group of 3 at the 1000s level, and three at the 100s level, none at the tens level, and two at the 1s level.



This shows that 3 goes into 3906 one thousand, three hundreds and two ones times. That is,

$$3906 \div 3 = 1302$$

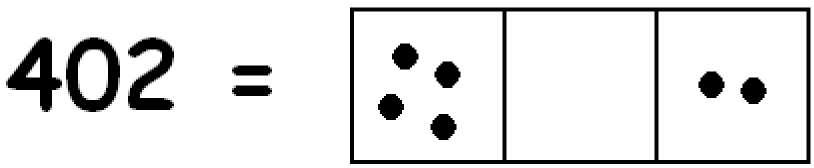
WEIRD LANGUAGE: The division sign $\div$ has an unusual name. It is called an <u>obelus</u>. Not many people know this.

Question 1: Draw a dots and boxes picture of 20486. Use your picture to show why $20486 \div 2$ equals 10243.

Let's try a harder one! Consider:

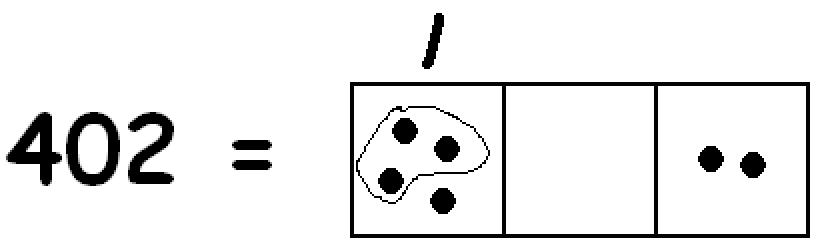
$$402 \div 3$$

Here's the picture:



and we are looking for groups of three dots: ●●●.

There is certainly one group at the 100s level.



and now it seems we are stuck – there are no more groups of three!

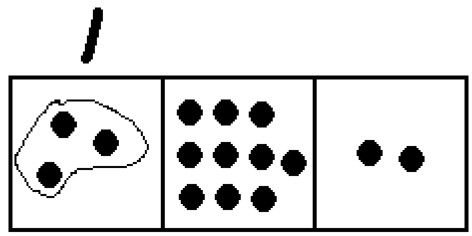
What can we do now? Are we really stuck?

Do you want a hint?

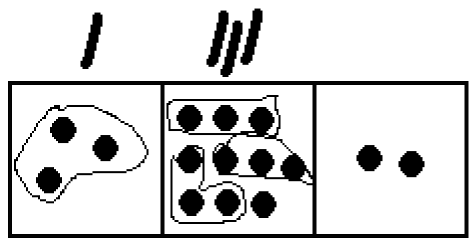
HINT: If our $1 \leftarrow 10$ rule means that each group of 10 becomes one in the box to the left, what might happen if we move one dot to the box to the right?

EPIPHANY: Let's <u>UN</u>EXPLODE a dot!

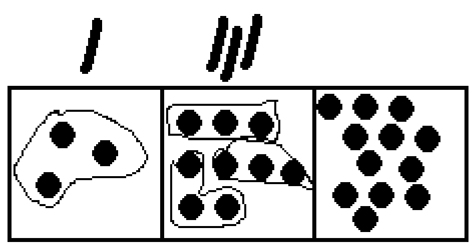
Since each dot is worth ten dots in the box to the right we can write …

402 = 

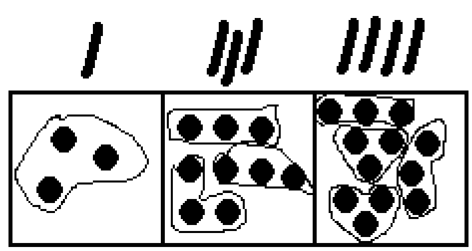
and find more groups of three:

402 = 

There is still a troublesome extra dot. Let's unexploded it too …

402 = 

giving us more groups of three:

402 = 

Finally we have the answer!

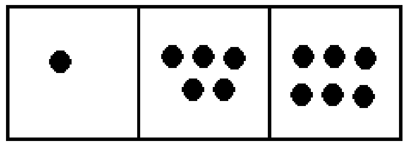
$$402 \div 3 = 134$$

Question 2: Compute $62124 \div 3$ via the dots and boxes method.

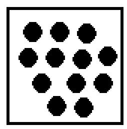
Question 3: Compute $61230 \div 5$ via the dots and boxes method.

Let's go up another notch of difficulty! Consider $156 \div 12$.

Here we are looking for groups of 12 in the picture …

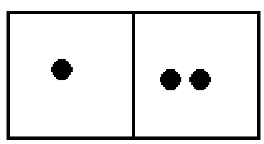
156 = 

What does 12 look like? It can be twelve dots in a single box:

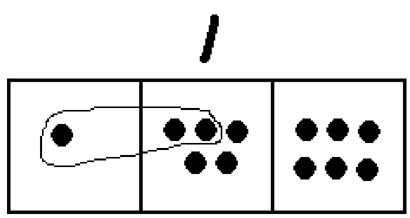
12 = 

<u>Question</u>: Are there twelve dots in the picture for 156? Does this mean that there is just one group of twelve in 156?

We don't' see any of those twelve single dots, but twelve can also be the result of an explosion:

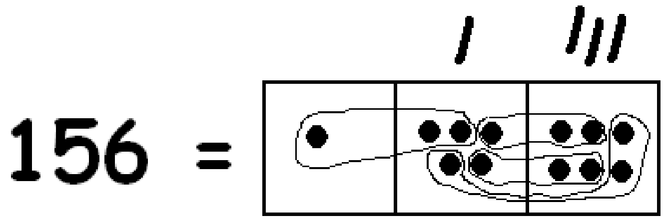
12 = 

and we certainly see some of these in the picture. There is certainly one at the tens level …

156 = 

(REMEMBER: With an unexplosion this would be twelve dots in the tens box.)

And three at the ones level:



We see:

$$156 \div 12 = 13$$

Question 4: Compute $13453 \div 11$ via the dots and boxes method.

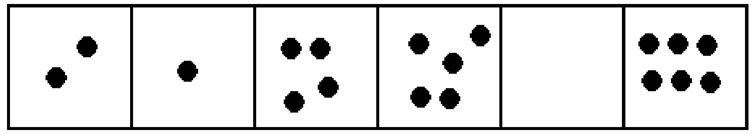
Question 5: Compute $4853 \div 23$ via this method.

Here's a bigger example:

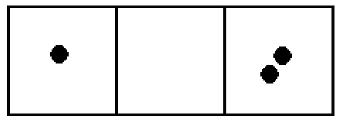
What's $214506 \div 102$?

STOP AND CONSIDER: Try this before continuing on. What does the 102 dots look like?

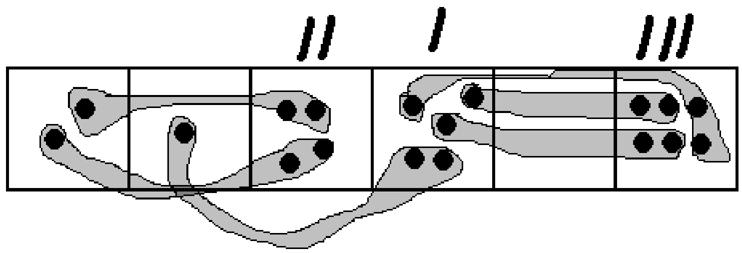
Here goes:

214506 = 

and we are looking for groups of 102 in this picture. These look like:

102 = 

We can do it!

214506 = 

$$214506 \div 102 = 2103$$

Question 6: Compute the following using dots and boxes.

$64212 \div 3$

$44793 \div 21$

$6182 \div 11$

$99916131 \div 31$

$637824 \div 302$

$2125122 \div 1011$

AHA MOMENT:

Question 7: Use dots and boxes to compute

$$2130 \div 10$$

Use dots and boxes to compute

$$41300 \div 100$$

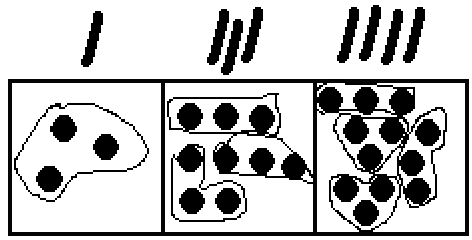
REFLECTION QUESTIONS: Look back at the previous two problems. What pictures did you use for 10 and for 100? Can you describe in words what happens when dividing by 10 and by 100?

B. WHAT TEACHERS TEACH

We used dots and boxes to show …

$$402 \div 3 = 134$$

402 =



Some teachers have their students solve this division problem by using a diagram like the following:

$$\begin{array}{r} 134 \\ 3\overline{)402} \\ \underline{3} \\ 10 \\ \underline{9} \\ 12 \\ \underline{12} \\ 0 \end{array}$$

At first glance this seems very mysterious, but it is really no different from the dots and boxes method. Here is what the table means.

To compute $402 \div 3$ students first make a big estimation as to how many groups of 3 there are in 402. Let's guess that there are 100 groups of three.

$$\begin{array}{rc} & \text{Groups of 3} \\ 3\overline{)402} & 100 \\ 300 & \end{array}$$

How much is left over after taking away 100 groups of 3?

	Groups of 3
3)402	100
300	
102	

How many groups of 3 are in 102? Let's try 30:

	Groups of 3
3)402	100
300	
102	
90	30

How many are left? There are 12 left and there are four groups of 3 in 12.

	Groups of 3
3)402	100
300	
102	
90	30
12	
12	4
0	

The accounts for entire number 402. And where doe we find the final answer? Just add the total count of groups of three that we tallied:

$$402 \div 3 = 100 + 30 + 4 = 134$$

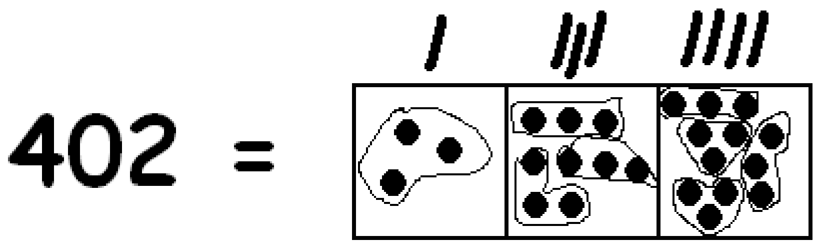
REFLECTION: Compare the two tables we've listed:

	Groups of 3
3)402	100
300	
102	
90	30
12	
12	4
0	

```
   134
3 )402
   3
   10
    9
    12
    12
     0
```

In what way are they the same? In what way are they different?

Look at the dots and boxes method:



In what way is the same or different from the two tables?

WHY DO TEACHERS LIKE THEIR METHOD? Because it is quick, not too much to write down, and it works.

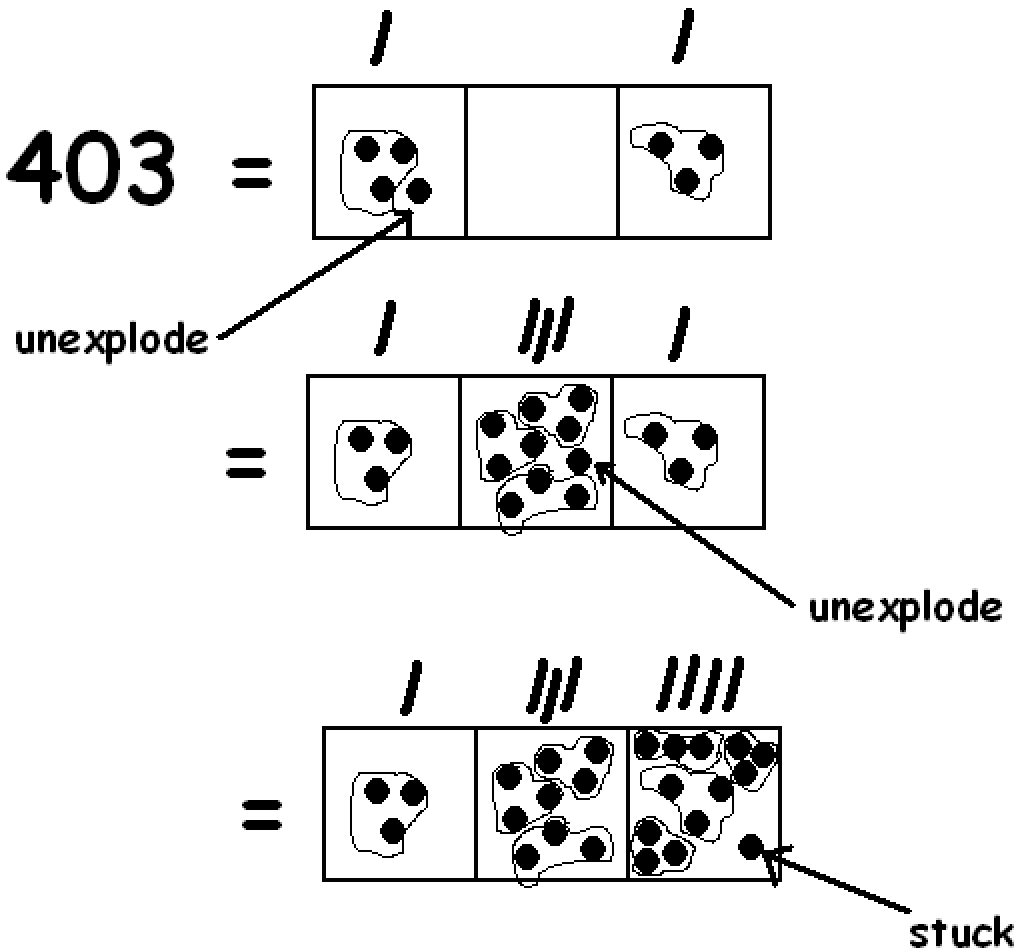
WHY DO WE LIKE THE DOTS AND BOXES METHOD? Because it easy to understand. (And drawing dots and boxes is kind of fun!)

C. WHEN PROBLEMS DON"T WORK OUT NICELY

We saw that 402 is evenly divisible by 3:

$$402 \div 3 = 134$$

This means that 403, one more, shouldn't be divisible by three. It should be one dot too big. Do we see the extra dot if we try the dots and boxes method?



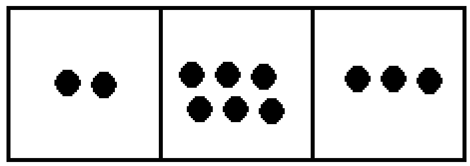
Yes we do! We have a remainder of one dot that can't be divided. We say that we have a <u>remainder</u> of one and some people like to write:

$$403 \div 3 = 134 \ R\ 1$$

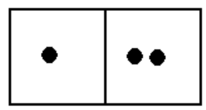
Let's try another one:

$$263 \div 12$$

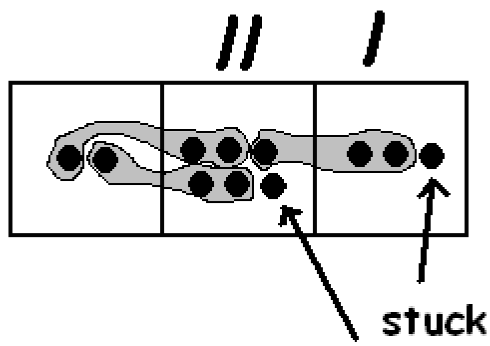
Here's what we have …

263 = 

and we are looking for …

12 = 

Here goes!

263 = 

Unexploding won't help any further and we are indeed left with one remaining dot in the tens position and a dot in the ones position we cannot divide. This means we have a remainder of … eleven.

$$263 \div 12 = 21R11$$

Question 8: Use dots and boxes to show that $5210 \div 4$ is 1302 with a remainder of 2.

Question 9: Use dots and boxes to compute $4857 \div 23$.

Question 10: Use dots and boxes to show that $31533 \div 101$ equals 312 with a remainder of 21.

Question 11: Recall that teachers teach their students to work compute $403 \div 3$ as follows:

$$
\begin{array}{r}
134 \\
3\overline{)403} \\
\underline{3} \\
10 \\
\underline{9} \\
13 \\
\underline{12} \\
1
\end{array}
$$

$$403 \div 3 = 134\,R1$$

Use the teachers' method to compute

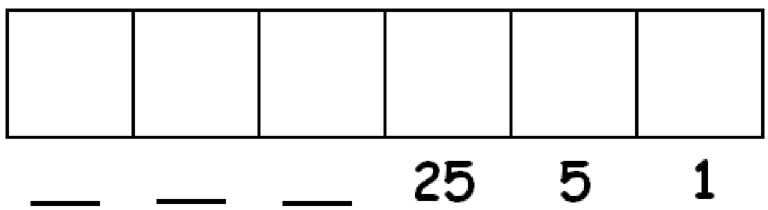
a) $5210 \div 4$
b) $4857 \div 23$
c) $31533 \div 101$

Which method do you like better: dots and boxes or the teachers' method? Or does it depend on the problem you are doing?

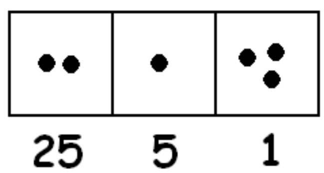
D. EXPLORATION AND ADVENTURE

All that we have done so far is in the every-day arithmetic of a $1 \leftarrow 10$ system. We could perform division in a $1 \leftarrow 5$ system as well if we liked.

ADVENTURE 1:
a) What is value of each box in a $1 \leftarrow 5$ system? Fill in the blanks:



b) The number 213 in the $1 \leftarrow 5$ system is really the number 58 in ordinary arithmetic.



$$25+25+5+1+1+1 = 58$$

What is the number 424 in the $1 \leftarrow 5$ system really in our arithmetic?
What is he number 11 in the $1 \leftarrow 5$ system really in our arithmetic?

c) Use the dots and boxes method to show that $424 \div 11$ equals 34 in the $1 \leftarrow 5$ system.

d) What is "34" in ordinary arithmetic? What is the statement $424 \div 11 = 34$ really in ordinary arithmetic? Is it indeed correct?

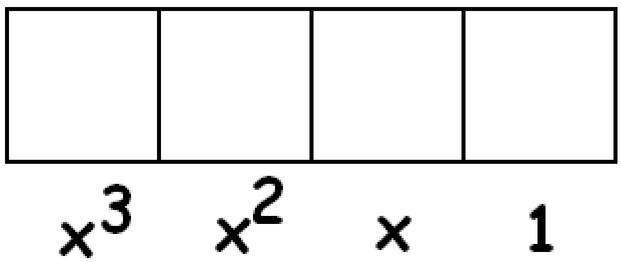
e) **TOUGH CHALLENGE:** Work out $2021 \div 12$ in the $1 \leftarrow 5$ system.
[WATCH OUT: You will need to unexplode. How many dots does an unexplosion become?]

ADVENTURE 2: Anu refuses to tell anyone if she is working in a $1 \leftarrow 10$ system, or a $1 \leftarrow 5$ system, or any other system. She makes everyone call it an $1 \leftarrow x$ system but won't tell a soul what number she has in mind for *x*.

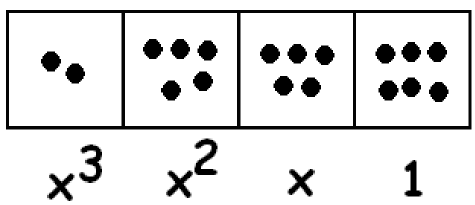
We know that boxes in a $1 \leftarrow 10$ have values that are powers of ten: 1, 10, 100, 1000, 10000, …

And boxes in a $1 \leftarrow 5$ system powers of five: 1, 5, 25, 125, 625, …

So Anu's system, whatever it is, must be powers of *x*:

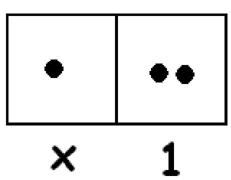


When Anu writes 2556 she must mean:



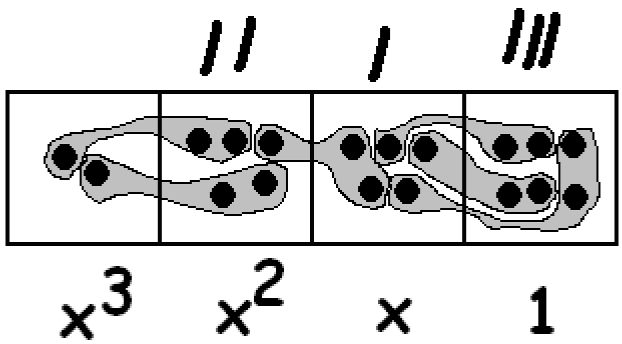
$$2x^3 + 5x^2 + 5x + 6$$

and when she writes 12 she means:



$$x + 2$$

Anu decides to compute $2556 \div 12$. She obtains:



$$\left(2x^3 + 5x^2 + 5x + 6\right) \div \left(x + 2\right) = 2x^2 + x + 3$$

Whoa!

a) Use Anu's method to find $\left(3x^2 + 7x + 2\right) \div \left(x + 2\right)$

b) Use Anu's method to find $\left(2x^4 + 3x^3 + 5x^2 + 4x + 1\right) \div \left(2x + 1\right)$

c) Use Anu's method to find $\left(x^4 + 3x^3 + 6x^2 + 5x + 3\right) \div \left(x^2 + x + 1\right)$

Anu later tells use that she really was thinking of a $1 \leftarrow 10$ system so that *x* does equal ten. Then her number 2556 really was two thousand, five hundred and fifty six and 12 really was twelve. Her statement:

$$\left(2x^3 + 5x^2 + 5x + 6\right) \div \left(x + 2\right) = 2x^2 + x + 3$$

is actually $2556 \div 12 = 213$.

d) What division problems did you actually solve for parts a, b and c in the $1 \leftarrow 10$ system?

HARD CHALLENGE: Uh Oh! Anu has changed her mind! She now says she was thinking of a $1 \leftarrow 11$ system!

Now "2556" means $2\times 11^3 + 5\times 11^2 + 5\times 11 + 6 = 3328$, "12" means $1\times 11 + 2 = 13$ and "213" means $2\times 11^2 + 1\times 11 + 3 = 256$, and so her computation

$$2556 \div 12 = 213$$

is actually the statement:

$$3328 \div 13 = 256$$

e) Check that this is still correct.

f) How do parts a, b and c translate for an $1 \leftarrow 11$ system?

ADEVNTURE 3: Show that $\left(x^4 + 4x^3 + 6x^2 + 4x + 1\right) \div \left(x+1\right)$ equals $x^3 + 3x^2 + 3x + 1$.

a) What is this saying for $x = 10$?
b) What is this saying for $x = 2$?
c) What is this saying *x* equal to each of 3, 4, 5, 6, 7, 8, 9, and 11?
d) What is it saying for $x = 0$?
e) What is it saying for $x = -1$?

MATHEMATICAL THINKING!
Exploration 5

PILES AND HOLES

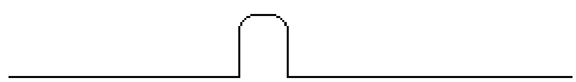
A. GETTING STARTED

Here is a story that isn't true.

When I was a young child I used to spend my days sitting in a sandbox at the back of my yard (not true). And being a very serene child I used to take my time in the morning leveling the sand in my box to make a perfectly flat horizontal surface (also not true). It very much appealed to my tranquil sensibilities, so much so that I decided to give this level state a name. I called it "zero."

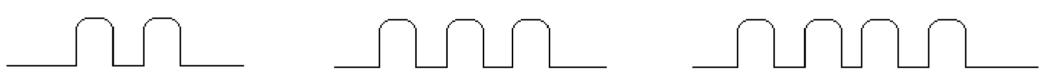
0

I spent many an hour admiring my zero state (still not true), but then, one day, I had an epiphany! I realized I could reach behind where I was sitting, grab a handful of sand and make a pile. I called the one pile the "1" state.



1

And then I discovered two piles – which I called "2" – and three piles, "3" and so forth.

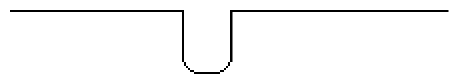


2 **3** **4** **etc.**

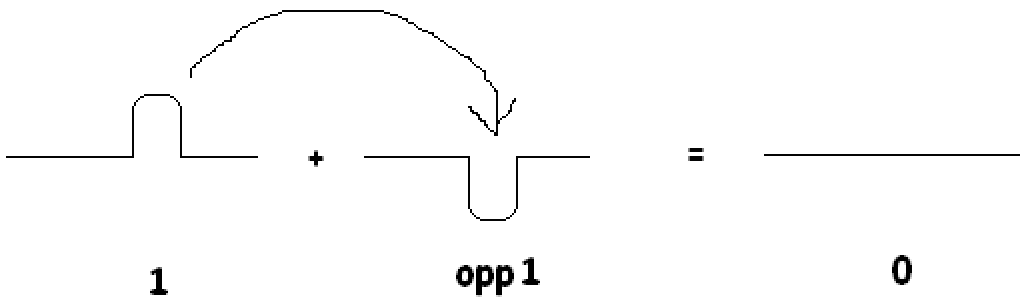
Hours of mathematical fun were had as I discovered the counting numbers with piles of sand.

Question 1: Had I discovered the counting numbers in this untrue story? Is counting piles of sand really the same as counting dots?

BUT THEN … one day I had the most astounding epiphany of all! Instead of using a handful of sand to make a pile, I could, I realised, take away a handful of sand and make the OPPOSITE of a pile, namely, a hole!



I called this "*opp* 1" for the opposite of one pile. And notice that "opposite," in some sense, is really the right word because a hole "cancels" a pile:



$$1 + opp1 = 0$$

Question 2: Explain what I am doing in this picture.

In the same way "*opp* 2" represents the opposite of two piles, namely, two holes, and "*opp* 5" the opposite of five piles, namely, five holes.

Question 3: Draw a picture for $2 + opp\ 2$ and draw its answer.

Question 4:
a) JinSe says that $3 + opp\,2$ equals one pile? Is she correct? Draw a picture.
b) Harold says $5 + opp\,7$ equals 2 holes. Is he correct? Draw a picture.

Question 5: a) What is the opposite of a hole?

b) What is the opposite of the opposite of the opposite of three piles?

c) What is the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of the opposite of ten hundred holes?

B. WHAT THE REST OF THE WORLD WRITES AND THINKS

As a young child I had, allegedly, discovered a whole host of new numbers, the opposites of the counting numbers. The world calls these the <u>negative numbers</u>. And instead of writing *opp* for opposite, people use a tiny dash "-".

So …

2 = two piles

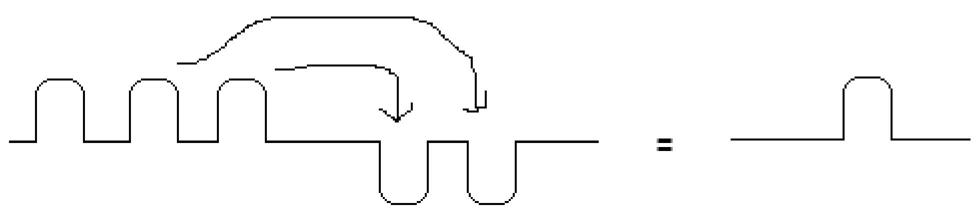
-2 = opposite of two piles
= two holes

--2 = opp opp 2 = two piles

-1 = one hole

-----------7 = seven holes

Unfortunately, matters become a little confusing when one starts combining piles and holes. For example, three piles and two holes together (giving one pile) is written:



3 + -2 = 1

and

$$5 + -7 \;= -2$$

is a clunky way of writing "five piles and seven holes equals two holes."

Question 6: a) Ali invented the notation "$4P+5H+3H+2P=2H$". What do you think he means by this? Do you like his notation?

b) Cuthbert writes $\overline{3}+\underline{4}+\overline{5}=\overline{4}$. What do you think he means by his notation?

Question 7: The statement:

$$-3+7=4$$

reads "3 holes and 7 piles makes 4 piles," and the statement:

$$17+-6+-4+6+-20=-7$$

reads "17 piles and 6 holes and 4 holes and 6 piles and 20 holes makes 7 holes."

Translate each of the following, and give the answers!

a) $5+-9+2$

b) $3+-10+11-5$

c) $2+-2+2+-2+2+-2$

d) $-6+-1+-2+-3$

Question 8: Pandi writes:

$$------5+--2+----3+-------8$$

This actually makes sense! What does it mean and what is the answer?

To be honest most teachers reading the previous two pages would be quite upset! They will never let students write:

$$5+-7 = -2$$

for instance. They prefer to invent a new idea and call it "subtraction."

YOU HAVE BEEN TAUGHT …

To read

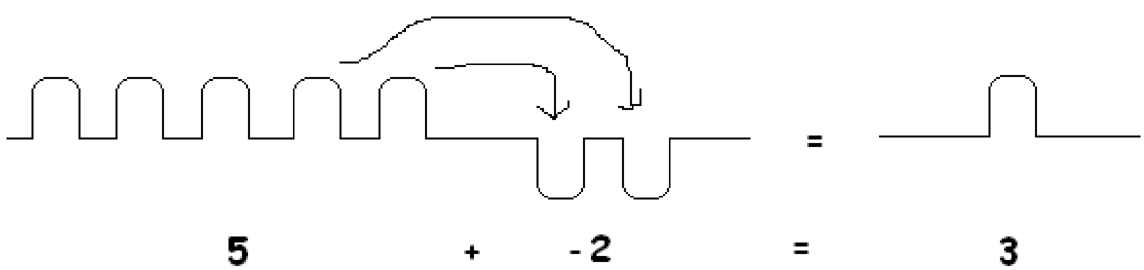
$$5-2$$

as "five take away two," which, of course, is three.

It is really

$$5+-2$$

"Five piles plus the ADDITION of two holes," which, of course, is three piles.



YOU HAVE BEEN TAUGHT …

To read:

$$6-4+1$$

as "six take away four plus one," which is three.

It is really

$$6+-4+1$$

"Six piles PLUS four holes PLUS one pile" which is three piles.

Question 9: Young children are usually taught to think "subtraction" and are often asked to compute problems like the following:

$$4-1=??$$
$$10-7=??$$
$$8-5=??$$

This is good for practicing skills. But often children will naturally ask about problems of the following type:

$$3-5=??$$
$$1-4=??$$
$$66-103=??$$

and sometimes they are told that these problems do not have answers.
Is this a correct thing to say or do they indeed have answers?

Do you think if young children were taught about piles and holes they could understand how to answer questions like these?

To a mathematician … subtraction doesn't really exist. It is just the ADDITION of opposites.

C. GROUPS OF PILES AND HOLES

Here's a vague question:

What is the opposite of three piles and two holes?

Does this mean …

The opposite of three piles AND THEN the addition of two holes

or

The opposite of both three piles and of two holes?

Question 10: One of these interpretations has the answer "five holes" and the other interpretation the answer "one hole." Which answer belongs to which interpretation?

Mathematicians use parentheses to help clarify such confusion. Parentheses group objects together. Some examples will explain:

EXAMPLE 1: $-(3+-2)$

This means the opposite of everything in the parentheses, the opposite of BOTH three piles and two holes. And what is the opposite of both three piles and two holes? Clearly, three holes and two piles!!

$$-(3+-2)=-3+2$$

This is one hole.

$$-(3+-2)=-3+2=-1$$

Comment: Most people write $-(3-2)=-3+2=-1$. Do you see the slight difference?

EXAMPLE 2: $-(3+-4+2)+5$

This is the opposite of all of 3 piles, 4 holes and 2 piles, AND THEN the addition of 5 piles. So we have 3 holes, 4 piles, 2 holes, and 5 piles. This is, in all, 3 piles.

$$-(3+-4+2)+5=-3+4+-2+5=3$$

Comment: Most people write this as $-(3-4+2)+5=-3+4-2+5=3$. Do you see the slight difference?

EXAMPLE 3: $6-(5-2)$

This is six piles and the opposite of both 5 piles and 2 holes.

$$6-(5-2)=6+-5+2$$

This equals 3 piles.

$$6-(5-2)=6+-5+2=3$$

EXAMPLE 4: $(5-2+1)-(3-2)$

This is 5 piles, 2 holes and 1 pile all grouped together PLUS the opposite of both 3 piles and 2 holes.

$$(5-2+1)-(3-2)=5+-2+1+-3+2$$

This equals 3 piles.

EXAMPLE 5: $10+(5-8)$

This is 10 piles PLUS a group of 5 piles and 8 holes.
$10+(5-8)=10+5+-8=7$

COMMENT: Teachers might call these examples of "distributing the negative sign."

Question 11: Interpret and evaluate the following:

a) $6-(3-2)$

b) $4+(1-3)$

c) $(3-9-1)-(8+2)$

d) $(1+2-4)-(3+2-4)$

e) $(2-3)-(5-2-3+7)-(-9+8-1)+(-2-3)$

f) $(16-16)-(17-17)$

Question 12: What's $-x$ if:

a) x is seven piles?
b) x is seven holes?
c) x is 50?
d) x is -50?

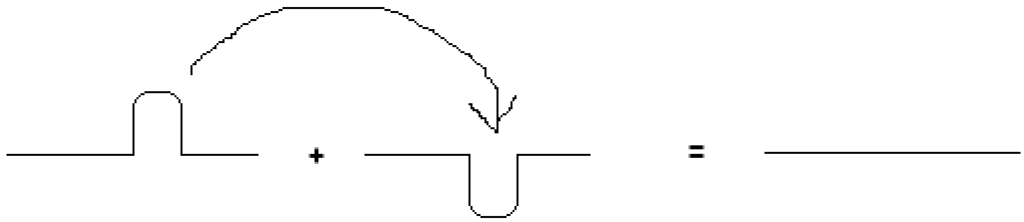
Question 13: CHALLENGE
Wanda is thinking of a number, which she calls W, but she refuses to tell us its value. But she does ask us to compute:

$$(4-W)-(2-W)$$

Without knowing her number do we have any chance of working this out?

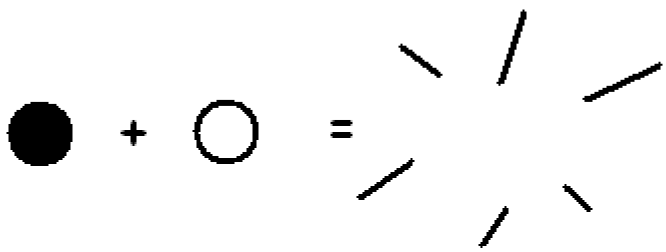
D. DOTS AND ANTI-DOTS

We have seen that the opposite of a pile is a hole. They "cancel" each other out!



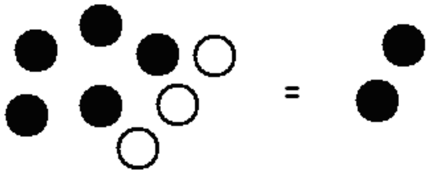
0

In the previous explorations we've been counting dots, not piles. So what's the opposite of a dot? Not sure. But whatever it is, let's call it an "anti-dot" and note, that like matter and antimatter, whenever a dot and anti-dot come together they both explode and leave nothing behind. We'll draw dots as solid dots and anti-dots as hollow circles:



$$1 + -1 = 0$$

So like piles and holes we can conduct basic arithmetic with dots and anti-dots:

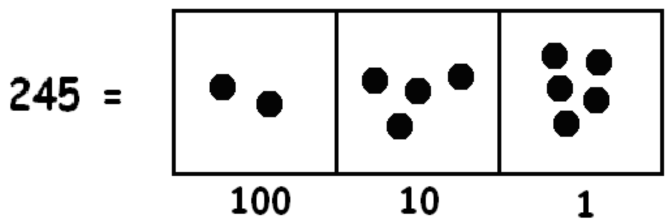


$$5 + -3 = 2$$

$$2 - 3 = -1$$

THE RETURN OF THE $1 \leftarrow 10$ MACHINE

Recall from exploration 2 that all numbers can be written using the powers of ten in a $1 \leftarrow 10$ machine. For example:

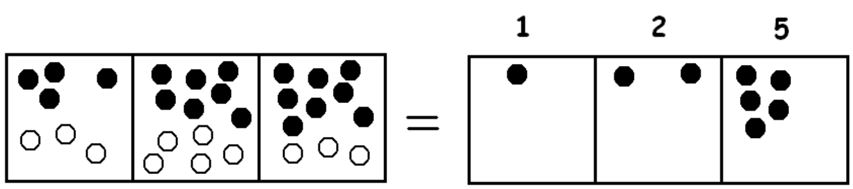


We have discovered in this exploration that subtraction is nothing but the addition of negative quantities. In our dots and boxes model let's work with dots and anti-dots.

Consider, for example, the problem:

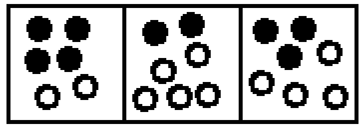
$$\begin{array}{r} 4\ 7\ 8 \\ -\ 3\ 5\ 3 \end{array}$$

This is an addition of dots and anti-dots problem:

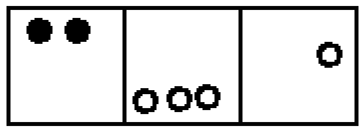


$$\begin{array}{r} 4\ 7\ 8 \\ -\ 3\ 5\ 3 \\ \hline =\ 1\ 2\ 5 \end{array}$$

As another example we see that 423 – 254, represented by diagram:



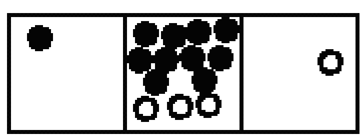
has answer:



That is:

$$\begin{array}{rccc} & 4 & 2 & 3 \\ - & 2 & 5 & 4 \\ \hline = & 2 & \vert -3 & \vert -1 \end{array}$$

This is absolutely valid mathematically, though the rest of the world may have difficulty understanding what "two hundred and negative thirty negative one" means! To translate this into familiar terms we can "unexplode" one of the solid dots to add ten solid dots to the middle box.



This gives the representation: 1 | 7 | -1. Unexploding again gives: 1|6|9. (Why?)

Thus we have:

423 – 254 = 169.

Question 14: How were you taught to answer this problem in school?

$$\begin{array}{r} 4\,2\,3 \\ -\ 2\,5\,4 \\ \hline = \quad\quad \end{array}$$

What is actually happening when you "borrow" a digit from another column?

Question 15: Compute each of the following two ways. First, the school way by starting at the right and borrowing digits. Then, do it by starting at the left and moving to the right, using negative numbers in the answer.

$$\begin{array}{r} 6\,3\,2\,8 \\ -\ 4\,4\,6\,9 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 7\,8\,3\,9\,0\,2\,3\,1 \\ -\ 3\,2\,4\,9\,5\,8\,4\,6 \\ \hline \end{array}$$

E. EXPLORATION AND ADVENTURE

ADVENTURE 1: CARD PILE TRICK

a) Take 10 red cards and 10 black cards from a deck of cards. Shuffle your 20 cards and arbitrarily split them into two equal piles. Count the number of red cards in the left pile and the number of black cards in the right pile. What do you notice? Repeat this activity two more times.

b) Shuffle your 20 cards and this time split them into a pile of 6 and a pile of 14 cards. Count the number of red cards in the small pile and count the number of black cards in the large pile. Take the (positive) difference of those two counts. Did you get 4? Repeat this two more times.

c) Shuffle the 20 cards again and this time split them into a pile of 9 cards and a pile of 11 cards. Count the number of red cards in the small pile, count the number of black cards in the large pile and take the (positive) difference of this count. What did you get? Repeat two more times. What do you notice?

d) Complete the following table:

Small Pile	Large Pile	Difference of #R in small and #B in large
10	10	0
9	11	
8	12	
7	13	
6	14	4
5	15	
4	16	
3	17	
2	18	
1	19	
0	20	

Any patterns?

e) Suppose, in a game with 5 cards in the small pile and 15 cards in the large pile, I counted three red cards in the small pile. Complete the following table:

	Small Pile 5	Large Pile 15
# reds	3	
# blacks		

What is the difference of counts of red cards in the small pile and black cards in the large pile?

f) **CHALLENGE:** Suppose the small pile again has 5 cards and the large pile has 15 cards. Suppose it turns out there are R red cards in the small pile. Complete the following table as an abstract exercise, writing formulas in each cell of the table:

	Small Pile 5	Large Pile 15
# reds	R	
# blacks		

What can you say about the difference between the number of red cards in the small pile and the number of black cards in the large pile?

Does this match what you observed in part d) of this question?

g) **OPTIONAL ULTIMATE CHALLENGE:** Call the number of cards in the small pile P and the number of cards in the large pile $20-P$. (Why is this the correct formula for this?) Suppose there are R red cards in the small pile. Complete the table again as the ultimate abstract exercise, writing formulas in each cell of the table:

	Small Pile P	Large Pile 20 - P
# reds	R	
# blacks		

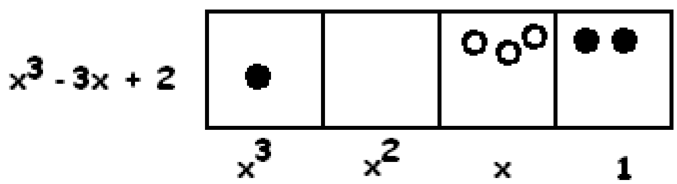
Can you write a formula for the difference of counts of the red cards in the small pile and black cards in the large pile? Does this formula match the data you obtained in part d) of this question?

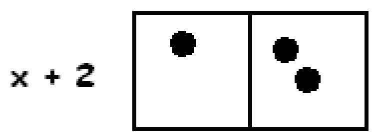
TOUGH ADVENTURE 2: DIVISION IN AN $1 \leftarrow x$ MACHINE

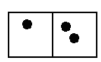
Recall that at the end of exploration 3 (if you chose to go that far!) we were completing division problems in a general $1 \leftarrow x$ machine. If you noticed, we only presented there examples with positive numbers. What about examples involving negative numbers?

Let's compute $\left(x^3 - 3x + 2\right) \div \left(x + 2\right)$ using dots and anti-dots.

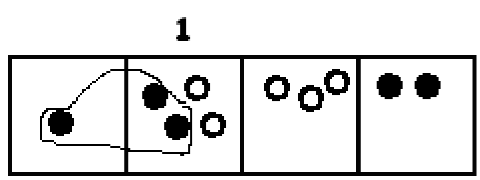
To do this, begin by drawing the representations of each quantity.



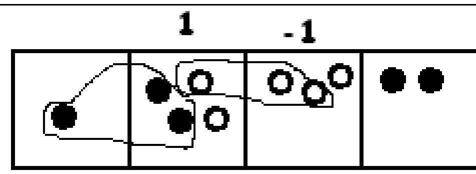


Our task is to find groups of  within the top diagram, and right away matters seem problematic. One might think to "unexplode" dots to introduce new dots (or anti-dots) into the diagram but there is a problem with this: We do not know the value of x and therefore do not know the number of dots to draw for each "unexplosion."

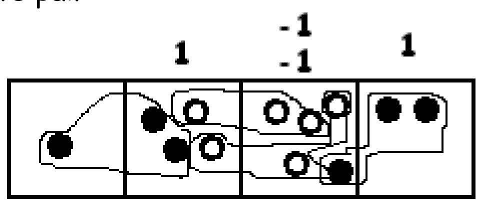
The way to cope with this difficulty is to employ an alternative trick: We can fill empty cells with dots and anti-dots. This keeps the value of the cell zero, but creates the dots we seek in our patterns.



It also creates anti-versions of what we seek:



Add another dot/antidote pair:



and we have:

$$\left(x^3-3x+2\right)\div\left(x+2\right)=x^2-2x+1.$$

Use dots-and-boxes to compute the following:

a) $(x^3-3x^2+3x-1)\div\left(x-1\right)$

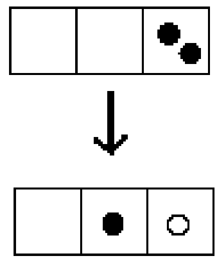
b) $\left(4x^3-14x^2+14x-3\right)\div\left(2x-3\right)$

c) $\left(4x^5-2x^4+7x^3-4x^2+6x-1\right)\div\left(x^2-x+1\right)$

d) $\left(x^{10}-1\right)\div\left(x^2-1\right)$

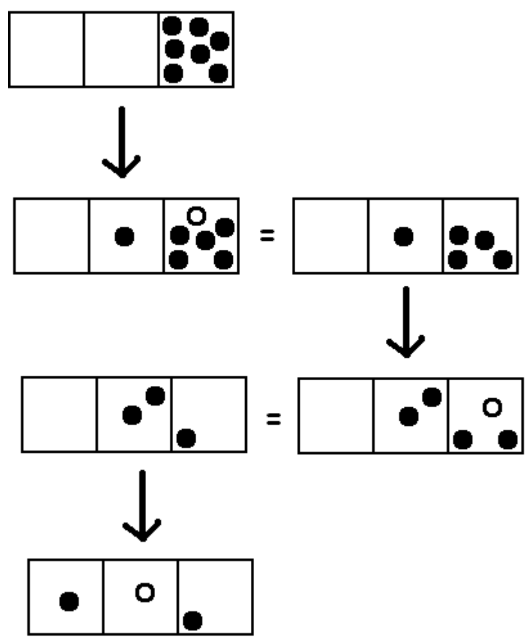
ADVENTURE 3: AN ANUSUAL MACHINE

Consider the $1\,|-1 \leftarrow 2$ machine which operates by exploding any two dots in one box and replacing them with an antidot in their box along with an actual dot one place to the left.



Putting seven dots into this machine, for example, produces the code:

$$1|-1|1$$

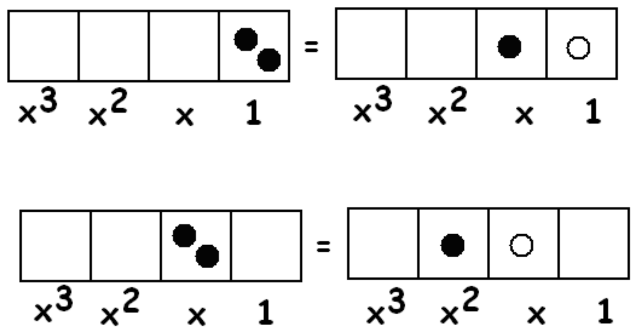


a) What is the code for 5 in this machine?

b) What is the code for 20 in this machine?

c) What number has code $1\,|\,0\,|\,0\,|-1$?

d) Fraviana says that there is a value of x that is "appropriate" for this machine:



etc

She's right! What is the "appropriate" value of x?

MATHEMATICAL THINKING!
Exploration 8

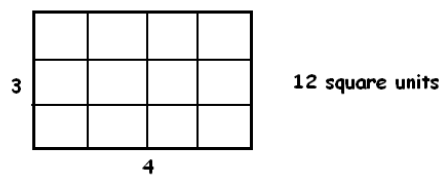
EXCERPTS

MULTIPLICATION

A. GETTING STARTED

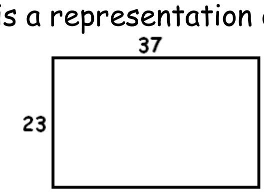
We've been counting dots. Let's now count squares.

Here are $3 \times 4 = 12$ squares:



This gives a picture of a 3-by-4 rectangle and subdivided into 12 squares units. (Maybe the units are inches or meters, Smoots or light-years. It doesn't matter for what we want to do in this exploration.) We like to call 3×4 the <u>area</u> of the rectangle. Thus we have a geometric interpretation of multiplication: The product of two numbers corresponds to the area of a rectangle.

For example, this next picture is a representation of the computation 23×37:



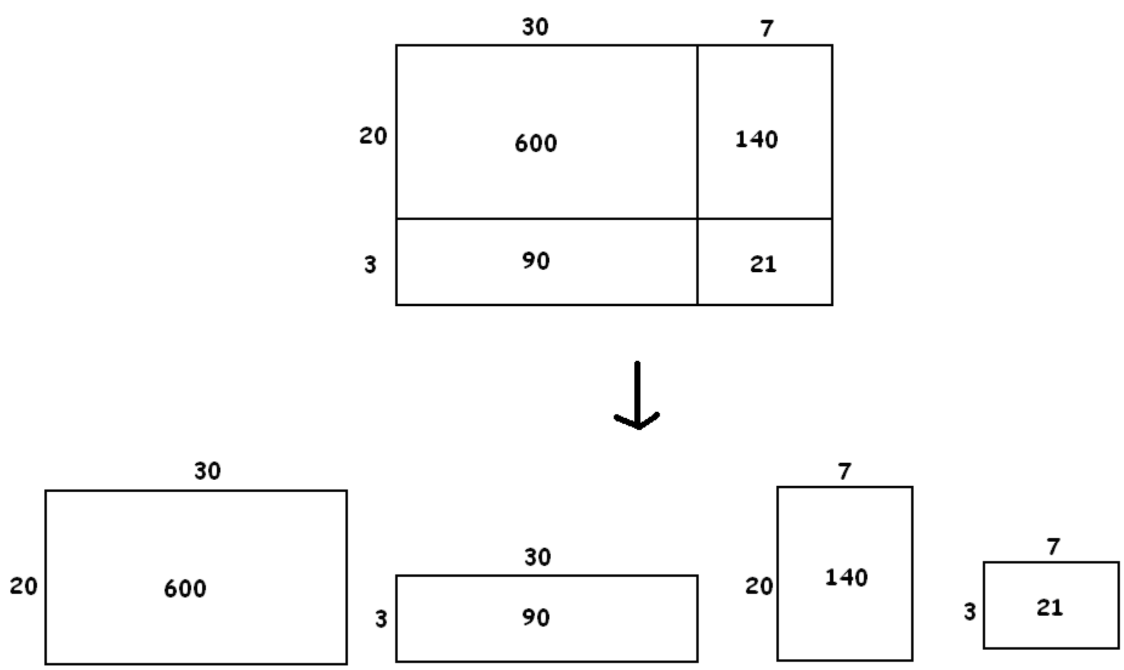
We can use this area interpretation to our advantage!

Comment: Mathematicians are somewhat coy about always regarding multiplication as area. Here we are working with positive whole numbers, and so it is easy to see that whole numbers of unit squares fit inside each rectangle. But what if the numbers aren't as nice as this? For example, does $3 \times (-4)$ correspond to the area of a rectangle? Does $\sqrt{32} \times 17\frac{3}{4}$? Maybe we should answer no, or maybe we should answer yes, or maybe we should adopt a different approach. This is what mathematicians do:

Let's work with positive whole numbers and see how area and multiplication should work for them. Then, we can decide whether or not then we would like to believe the same ideas should hold for all types of numbers.

So even though $3\times(-4)$ might not correspond to the area of an actual rectangle, maybe we can decide that numbers should work the same way as though it did! Let's see how area can help us compute 23×37. The numbers 23 and 37, you might agree, are awkward to work with directly.

But we can simplify matters by breaking the "23" into two smaller numbers that are easier to work with, namely 20 + 3, and the "37" into 30 + 7. This corresponds to subdividing the rectangle into four pieces as shown:



The area of the entire rectangle is just the sum of the area of these four pieces. We have:

$$23\times 37 = 600+140+90+21 = 851$$

This is easy to compute! One can almost do it in one's head.

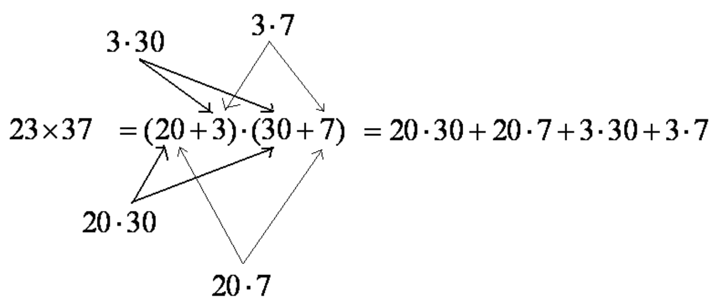
Comment: Why did we choose $23 = 20+3$ 2 and $37 = 30+7$, and not $23 = 16+7$ and $37 = 18+15$, say? Because, as we saw in exploration 7, multiplying multiples of ten is fairly straightforward!

Notice what we did here. We wrote:

$$23\times 37 \quad = (20+3)\cdot(30+7)$$

$$= 20\cdot 30 + 20\cdot 7 + 3\cdot 30 + 3\cdot 7$$

This amounts to selecting one number from the first set of parentheses, one from the second set, multiplying, and adding, making sure that each possible combination of one number from each set of parentheses is covered.



Some more examples:

EXAMPLE: Compute 15×17

Answer:

	10	7
10	100	70
5	50	35

$$15 \times 17 = (10+5)\cdot(10+7)$$

$$= 10\cdot10 + 10\cdot7 + 5\cdot10 + 5\cdot7$$

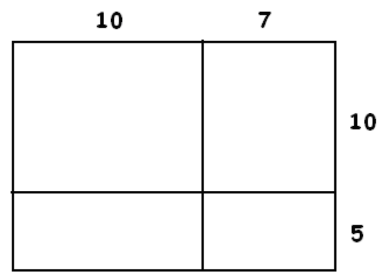
$$= 100 + 70 + 50 + 35$$

$$= 255$$

Again, for the expression $(10+5)\cdot(10+7)$, we ended up simply selecting one term from each set of parentheses and working with all possible combinations.

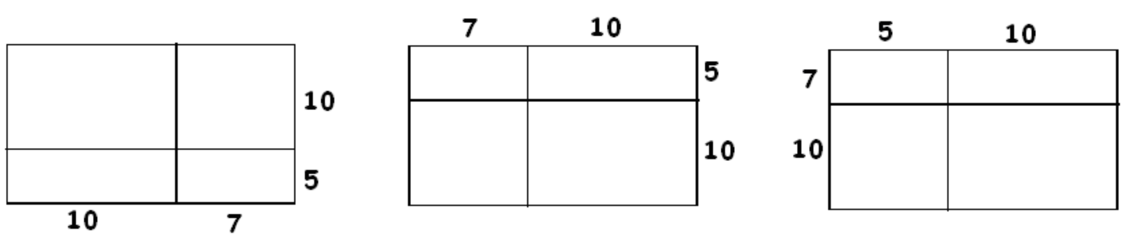
Question 1:

a) In computing 15×17 Vera started by drawing and labeling a rectangle this way:

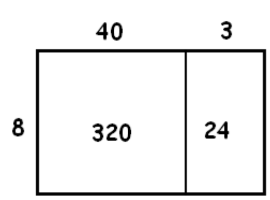


Is this a valid approach or is she wrong to label this way?

b) Dan, Cuthbert and Edna draw and label rectangles each of these ways. Are their approaches also valid?



EXAMPLE: Compute 8×43



$$8 \times 43 = (8) \times (40+3)$$

$$= 8 \times 40 + 8 \times 3$$

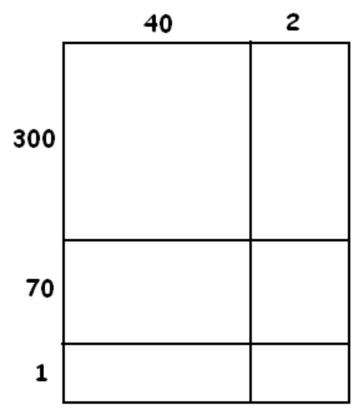
$$= 320 + 24$$

$$= 344$$

Even here, in "expanding" $(8)\times(40+3)$, we ended up selecting one term from each set of parentheses and listing all possible combinations.

Example: Compute 371×42

Answer: This corresponds to a rectangle subdivided into six pieces



$$371 \times 42 = (300+70+1)\cdot(40+2)$$

$$= 300\cdot 40 + 70\cdot 40 + 1\cdot 40 + 300\cdot 2 + 70\cdot 2 + 1\cdot 2$$

$$= 12000 + 2800 + 40 + 600 + 140 + 2$$

$$= 15582$$

And here too, in "expanding" $(300+70+1)\cdot(40+2)$ we simply ended up using each term of the first set of parentheses and multiplying it with each term of the second set of parentheses.

Example: Compute 23×37 as $(10+10+3)\times(10+10+10+7)$.

Answer: This is a crazy approach, but the mathematics still works!

	10	10	10	7
10	100	100	100	70
10	100	100	100	70
3	30	30	30	21

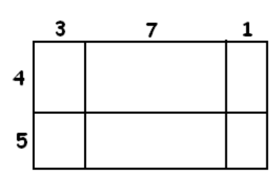
$$23 \times 37 = 100+100+100+100+100+100+30+30+30+70+70+21 = 851$$

EXAMPLE: The computation:

$$(4+5)\cdot(3+7+1)$$

corresponds to subdividing a rectangle into how many pieces?

Answer:



Six pieces!

We have:

$$\begin{aligned}(4+5)\cdot(3+7+1) &= 4\cdot 3+4\cdot 7+4\cdot 1+5\cdot 3+5\cdot 7+5\cdot 1\\ &= 12+28+4+15+35+5\\ &= 99\end{aligned}$$

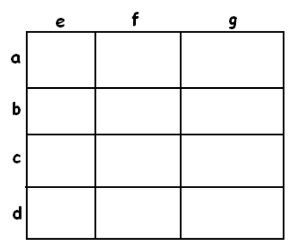
$\square$

Comment: This was a very complicated way of computing $9\times 11=99$!

EXAMPLE: What does this computation mean geometrically?

$$(a+b+c+d)\cdot(e+f+g)$$

Answer: This corresponds to subdividing a rectangle into 12 pieces.



With the appropriate patience, one could write this out:

$$(a+b+c+d)\cdot(e+f+g)=ae+af+ag+be+bf+bg+ce+cf+cg+de+df+dg \quad \square$$

Comment: Remember that mathematicians usually omit the multiplication sign "$\times$" or "$\cdot$" when multiplying two quantities represented by symbols.

Question 2: One can compute $(2+3)\cdot(7+4)$ two ways:

Short way: $(2+3)\cdot(7+4)=5\cdot 11=55$

Long way: $(2+3)\cdot(7+4)=2\cdot 7+2\cdot 4+3\cdot 7+3\cdot 4=14+8+21+12=55$

Compute each of the following both the short way AND the long way.

a) $(2+3)\cdot(3+7)$
b) $(2+4+6)\cdot(2+6+1+3)$

Question 3: We have, for example:

$$(a+b)(x+y)=ax+bx+ay+by$$

i) Expand the following expressions:

a) $(w+x)(z+r)$
b) $(a+x+b)(z+y)$

ii) If you were to expand the following expression, how many separate terms would it yield?

$$(x+y+z+w+t+r)(a+b+c+d+e+f+g+h)$$

Question 4: Here's how to calculate 13×26:

$$13\times 26=(10+3)(20+6)=200+60+60+18=338$$

Without a calculator, evaluate each of the following:

a) 23×14
b) 106×21
c) 213×31

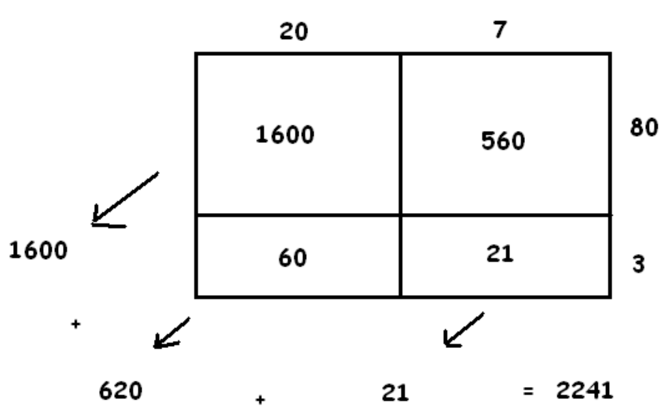
WHAT YOU MAY HAVE LEARNED IN SCHOOL …

How were you taught to compute 83×27, for example, in school? Were you taught to write something like the following?

$$\begin{array}{r} 83 \\ \times\ 27 \\ \hline 21 \\ 560 \\ 60 \\ 1600 \\ \hline 2241 \end{array}$$

This is really no different than the "expanding brackets" technique of this unit:

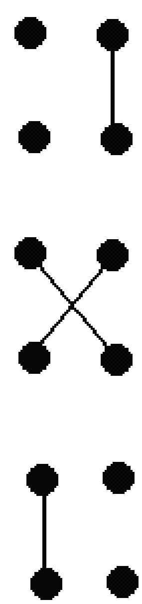
$$83 \times 27 = (80+3)(20+7) = 1600 + 60 + 560 + 21$$



Question 5: Do you see how the two approaches agree?

Notice that adding the numbers in the cells according to the diagonals of the picture conveniently groups the 100s, the 10s and the units.

Question 6: Many students in India are taught to memorize the following diagram for multiplying two two-digit numbers (such as 83×27).



What is this diagram prompting students to do?

Question 7: Nervous Nelly memorized the word FOIL for expanding expressions of the form $(x+y)(a+b)$. She would write:

$$(x+y)(a+b) = xa + xb + ya + yb$$

in that order. Here F stands for "first," O for "outer," I for "inner" and L for "last."

a) What do you think the words first, outer, inner and last are referring to?

b) Does one have to expand in this order? John likes to FLIO and Olivia likes to OLFI. Will they be incorrect?

c) How useful is it to memorize FOIL? What will Nervous Nelly do with $\left(x+y+z\right)\left(a+b+c+d+e+f\right)$?

We have the following "rule" for expanding brackets:

To compute the product of two sums, select one term from each set of parentheses, multiply, and sum the results. Make sure to list all possible combinations.

e.g. $(a+b+c)(x+y+w+z) = ax+bx+cx+ay+...$

e.g. $(a+b)\times c = ac+bc$

e.g. $(x+y)\cdot(p+q) = xp+yp+xq+yq$

Comment: The second example above can be thought of the product of two sums with parentheses. Think of it as: $(a+b)\times(c)$. Selecting one term from each set of parentheses corresponds to always choosing "*c*" in the second set. It leads to the expansion: $ac+bc$.

One also sees that $a\left(b+c\right) = ab+ac$.

People like to call this specific example of expanding brackets the <u>distributive property</u>.

Question 8: What picture of a rectangle goes with the distributive rule $a\left(b+c\right) = ab+ac$?

Comment: Are you perturbed by the fact that we mixed and matched multiplication symbols in the three examples at the top of this page? Mathematicians tend not to be strict about which symbol to use when, and will often simply decide to use what looks best visually. Usually "$\times$" is not used in algebra – it can be confused with the variable x - but we used it in "$(a+b)\times c = ac+bc$" because there is no danger of confusion there!

D. TAKING IT FURTHER …

If $(2+3)(4+5)$ corresponds to subdividing a rectangle (in two dimensions) into four pieces … What does

$$(2+3)\cdot(4+5)\cdot(6+7)$$

correspond to geometrically?

Now we are multiplying together three sums – three dimensions?

Here's a three-dimensional box. It is subdivided into eight pieces as according to the expression: $(2+3)\cdot(4+5)\cdot(6+7)$. (Think of a block of cheese.)

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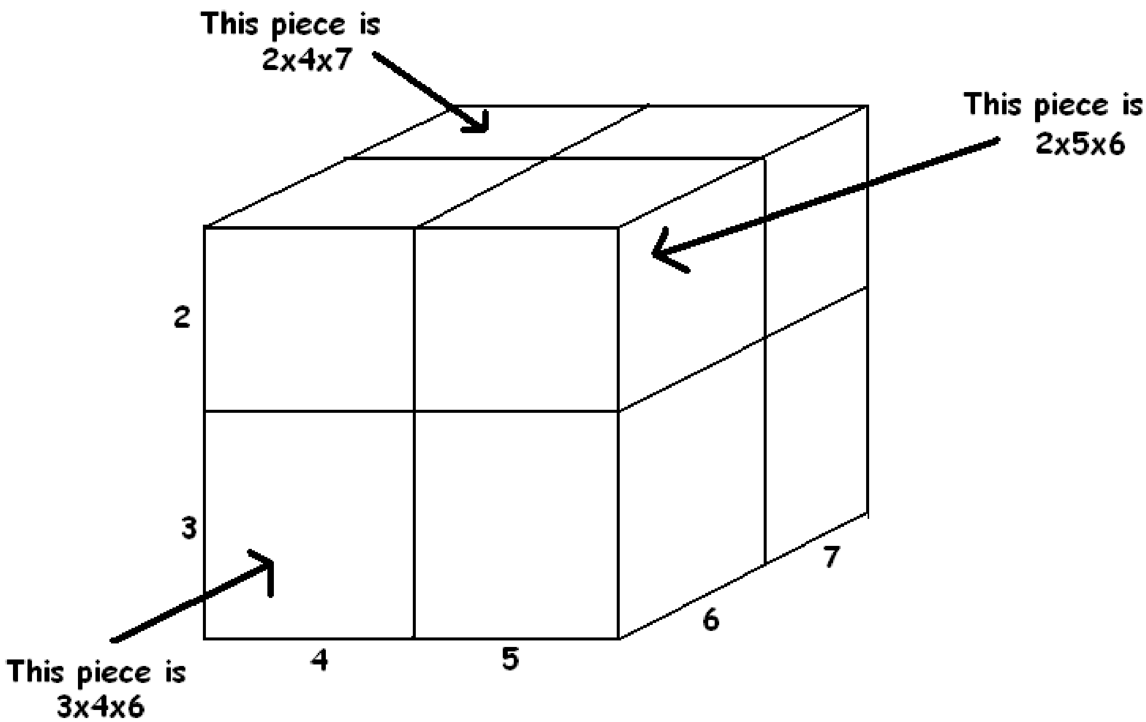


Question 9: Which piece is the $3\times5\times6$ piece? Which is the $3\times5\times7$ piece? What are the dimensions of the top left front piece? There is one piece completely out of view. Where is it and what are its dimensions?

Here are the eight pieces:

$$\begin{aligned}(2+3)\cdot(4+5)\cdot(6+7) = \; & 2\times4\times6 \\ & +2\times4\times7 \\ & +3\times4\times6 \\ & +3\times4\times7 \\ & +2\times5\times6 \\ & +2\times5\times7 \\ & +3\times5\times6 \\ & +3\times5\times7\end{aligned}$$

NOTICE AGAIN! This corresponds to selecting one term from each set of parentheses and making sure all possible combinations appear!

What we have actually shown in this example is that:

$$5 \cdot 9 \cdot 13 = 48 + 56 + 72 + 84 + 60 + 70 + 90 + 105 = 585$$

which is, indeed, correct.

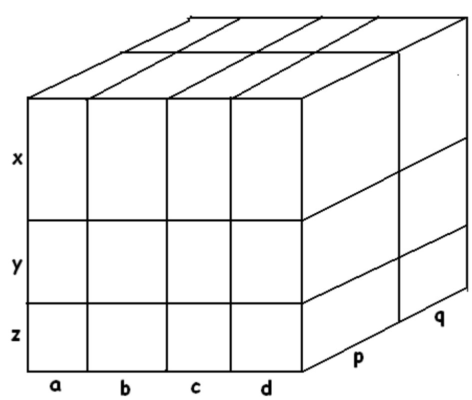
EXAMPLE: If one were to expand:

$$(x + y + z)(a + b + c + d)(r + s)$$

how many terms would there be? What does this correspond to geometrically?

Answer: There would be $3 \times 4 \times 2 = 24$ terms. (Do you see why?)

The expression corresponds to subdividing a rectangular box into 24 pieces.



Question 10: If one were to expand:

$$(x+y)(x+a+b)(a+c+p)$$

a) How many terms would there be?
b) Would xac be one of those terms? How about cay, xcp, xax, and xyc?

Question 11: If one were to expand:

$$(a+b+c+d+e)(w+x)(a+x+b+t+r)(e+f)$$

how many terms would there be?

(Is there a geometric interpretation to this expression?)

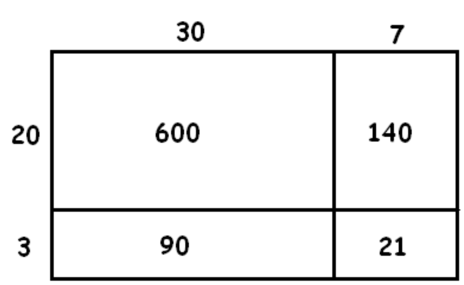
Comment: When a number represented by a symbol x is multiplied by itself we write x^2 and say "x squared." Notice the word <u>square</u> here. We say this because of the geometry: x^2 is the area of a square of side-length x.

And what do we say for $x^3 = x \times x \times x$? Answer: x <u>cubed</u>. This is because x^3 is the precisely volume of a cube of side-length x.

Question 12: Why have we no word for $x^4 = x \times x \times x \times x$?

F. MULTIPLICATION OF NEGATIVE NUMBERS

Recall that the computation 23×17, for example, computes the area of a 23-by-17 rectangle. By subdividing the rectangle into pieces we can find its area with ease
:



23 x 37 = 600 + 90 + 140 + 21 = 851

And recall from page 8 that mathematicians are a little coy about regarding linking multiplication with area. It can be interesting, for the sake of play, to decide that all the rules of multiplication we've developed so far should continue to hold even if the "area" idea starts to break down.

For example, let's now give ourselves permission to incorporate negative quantities into this model and compute 13×39 again, but a different (and strange) way.

	40	-1
10	400	-10
3	120	-3

13 x 39 = 400 +120 - 10 - 3 = 507

(Recall from exploration 4 that $3 \times (-1) = -1 + -1 + -1 = -3$ and $10 \times (-1) = -10$.)

The answer we obtained $13 \times 39 = 507$ is correct.

Although one can't have negative lengths and negative area in geometry, the algebra represented by these pictures is still the same and gives us answers that we feel we should believe.
Now let's try this again to discover a surprise!

Let's compute 18×17, for example, two different ways. First by representing 18×17 by $(10+8)(10+7)$ and then by $(20-2)(20-3)$:

	10	7
10	100	70
8	80	56

18 x 17 = 100 + 80 + 70 + 56
= 306

	20	-3
20	400	-60
-2	-40	??

18 x 17 = 400 - 60 - 40 + ??
= 306

We've left a question mark in the box for $(-2)\times(-3)$? What should a negative times negative number be?

We've all been told that it is positive, but is there a reason why this should be the case?

The example above indicates why:

> We need $400-60-40+??=306$. We have no choice. The question mark has to be POSTIVE six.

$$(-2)\times(-3)=6$$

In general:

If we want multiplication to work the same way for all numbers, then we have no logical choice to set negative times negative to be positive.

Question 13:

a) Doubtful Drew doesn't believe that $(-3)\times(-5)$ should be positive 15.

Can you compute 17×15 in two different ways (drawing a rectangle each time) and convince Doubtful Drew that the answer really should be 15?

b) How could you convince Doubtful Drew that $(-7)\times(-8)$ is positive 56?

Let's work out $(3-8)(2-5)$ two different ways:

Short Way: $(3-8)(2-5)=(-5)(-3)=15$

Long Way:

$$\begin{aligned}(3-8)(2-5) &= (3+-8)(2+-5)\\ &= 3\cdot 2+(-8)\cdot 2+3\cdot(-5)+(-8)\cdot(-5)\\ &= 6+-16\ +-15\ +40\\ &= 15\end{aligned}$$

Question 14: Work out each of the following two different ways: the short way and the long way.

a) $(4-6)\cdot(10-3)$
b) $(4+6)\cdot(10-3)$
c) $(4-6)\cdot(10+3)$
d) $(4+6)\cdot(10+3)$
e) $(4-7+3+2)(4+6-8+2-4)$

Question 15: Compute 25×25 three different ways: First as $(20+5)(20+5)$ and then as $(30-5)(30-5)$ and then as $(20+5)(30-5)$.

Question 16: a) If one were to expand $(x+y-a)(p+q-b)$, how many pieces would there be?

b) What is $(x+y-a)(p+q-b)$?

Some basic practice …

EXAMPLES: Simplify or evaluate:

a) $5\times(-3)$

b) $(-6)\cdot(-7)$

c) $(4-6)\cdot(10-3)$

d) $(-2)\cdot(-3+F)$

e) $(-3)\cdot(-5-G)$

f) $2-5\left(2-3T\right)$

Answers:

a) $5\times(-3)$ = -15

b) $(-6)\cdot(-7)$ = 42

c)Long way:

$$\begin{aligned}(4-6)(10-3) &= 4\cdot 10+(-6)\cdot 10+4\cdot(-3)+(-6)\cdot(-3)\\ &= 40-60-12+18\\ &= -14\end{aligned}$$

Short way:

$$\begin{aligned}(4-6)\cdot(10-3) &= (-2)\cdot 7\\ &= -14\end{aligned}$$

d) $$\begin{aligned}(-2)\cdot(-3+F) &= (-2)\cdot(-3)+(-2)\cdot F\\ &= 6-2F\end{aligned}$$

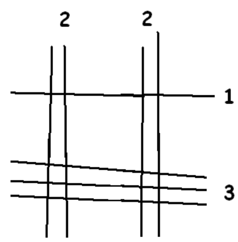
e) $$\begin{aligned}(-3)\cdot(-5-G) &= (-3)\cdot(-5)+(-3)\cdot(-G)\\ &= 15+3G\end{aligned}$$

f) $$\begin{aligned}2-5\left(2-3T\right) &= 2+\left(-5\right)\left(2+\left(-3\right)T\right)\\ &= 2+\left(-10\right)+15T\\ &= 15T-8\end{aligned}$$

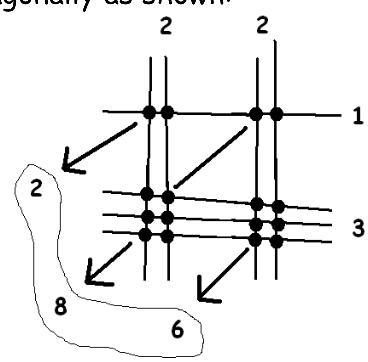
E. EXPLORATION

ADVENTURE 1: LINES AND INTERSECTIONS METHOD

Here's an unusual way to perform long multiplication. To compute 22×13, for example, draw two sets of vertical lines, the left set containing two lines and the right set two lines (for the digits in 22) and two sets of horizontal lines, the upper set containing one line and the lower set three (for the digits in 13).

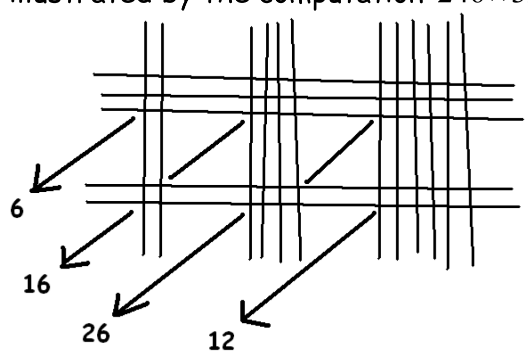


There are four sets of intersection points. Count the number of intersections in each and add the results diagonally as shown:



The answer 286 appears.

There is one caveat as illustrated by the computation 246×32:



Although the answer 6 thousands, 16 hundreds, 26 tens, and 12 ones is absolutely correct, one needs to carry digits and translate this as 7,872.

a) Compute 131×122 via this method.

b) Compute 54×1332 via this method.

c) How best should one compute 102×30054 via this method?

d) Why does the method work in general?

ADVENTURE 2: FINGER MULTIPLICATION

Here is an easy way to compute multiplications for your six- through ten-times tables. First encode numbers this way:

A closed fist represents "five" and any finger raised on that hand adds "one" to *that value.*

Thus a hand with two fingers raised, for example, represents "seven" and a hand with three fingers raised represents "eight." To multiply two numbers between five and ten, do the following:

1. Encode the two numbers, one on each hand, and count "ten" for each finger raised.

2. Count the number of unraised *fingers on each hand and multiply together the two counts.*

3. Add the results of steps one and two. This is the desired product.

For example, "seven times eight" is represented as two raised fingers on the left hand, three on the right hand. There are five raised fingers in all, yielding the number "50" for step one. The left hand has three lowered fingers and the right, two. We compute: $3 \times 2 = 6$. Thus the desired product is $50 + 6 = 56$. Similarly, "nine times seven" is computed as $60 + 1 \times 3 = 63$, and "nine times nine" as $80 + 1 \times 1 = 81$. Notice that one is never required to multiply two numbers greater than five!

CHALLENGE: Explain why this works.

(HINT: Suppose *a* fingers are raised on the left hand, and *b* on right. This means we are looking for the product $(5+a)(5+b)$. Write the algebraic expression that results from performing steps 1, 2, 3. Show that the two algebraic expressions are the same.)

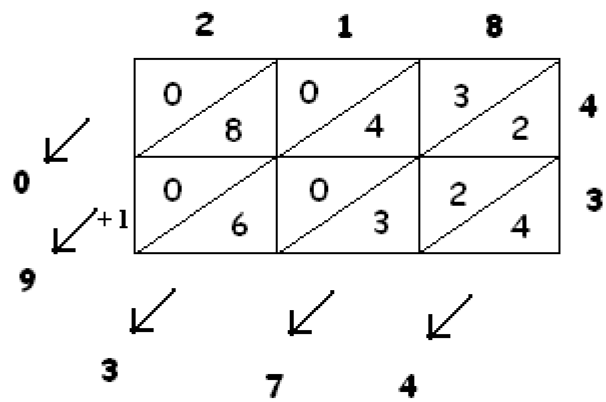
FINGERS AND TOES:
One can compute higher products using the same method! For example, with fingers and toes, one interprets 17×18 as "seven raised fingers" and "eight raised toes." This time we count each raised digit as "twenty" (we have twenty digits fingers and toes in all!) yielding: $17 \times 18 = 20 \times 15 + 3 \times 2 = 306$!

EXERCISE: Explain why this works!

ADVENTURE 3: LATTICE METHOD:

In the 1500s in England students were taught to compute long multiplication using following *galley method* (also known as the *lattice method* or the *Elizabethan method*):

To multiply 218 and 43, for example, draw a 2×3 grid of squares. Write the digits of the first number along the top of the grid and the digits of the second number along the right side. Divide each cell of the grid diagonally and write in the product of the column digit and row digit of that cell, separating the tens from the units across the diagonal of that cell. (If the product is a one digit answer, place a 0 in the tens place.)



Add the entries in each diagonal, carrying tens digits over to the next diagonal if necessary, to see the final answer. In our example, we have $218\times 43 = 9374$.

a) Compute 5763×345 via the galley method.

b) Explain why the galley method is really the "expanding brackets" technique $5763\times 345 = (5000+700+60+3)(300+40+5)$ in disguise. What is the specific function of the diagonal lines in the grid?

MATHEMATICAL THINKING!

Exploration 14

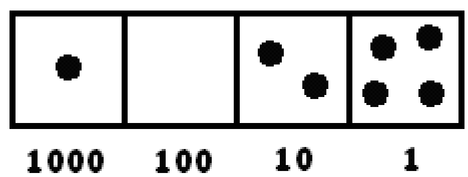
EXCERPTS

DECIMALS

A. GETTING STARTED

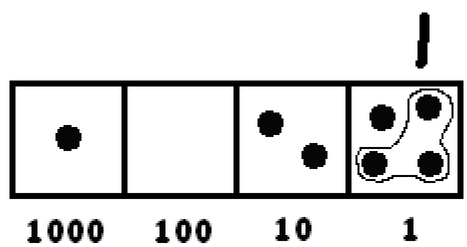
Recall from exploration 3 that a $1 \leftarrow 10$ machine consists of a row of boxes extending to the left, with the rightmost box representing units and each box thereafter representing 10 times the quantity of the box to its right. (Each cell of the machine thus represents a power of 10.)

1024 =



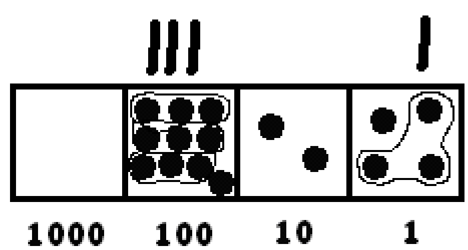
And recall from exploration 4 that we can perform division in a $1 \leftarrow 10$ machine by looking for groups of the desired pattern within the machine. For example, in the division problem $1024 \div 3$, we seek groups of three dots in the diagram above. One is evident:

1024 =



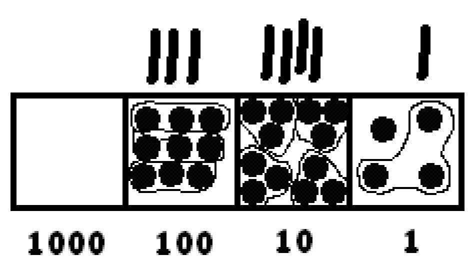
To see more, we must "unexplode" a dot:

1024 =



and again:

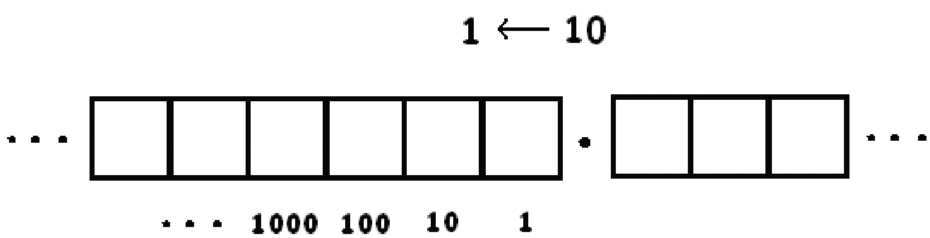
1024 =



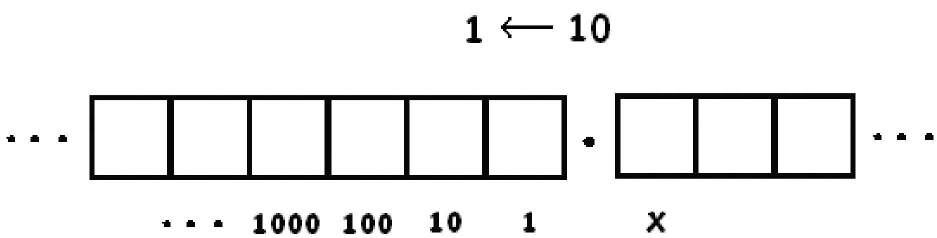
This leaves one obstinate dot remaining that can't be unexploded, or contended with in any clever way. We see:

$$1024 \div 3 = 341\ R\ 1$$

BUT … If we had additional boxes to the right we could unexplode this one obstinate dot and find further groups of three. But what would boxes further to the right represent? (It has become the convention to separate the boxes to the right with a point, called a <u>decimal point</u>.)



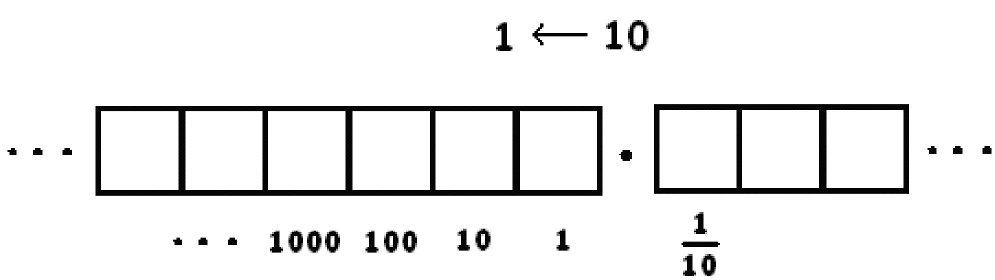
Suppose the first box to the right of the decimal point has value x.



Then, since this is a $1 \leftarrow 10$ machine, ten dots in this one box must be equivalent to one dot in the units box:

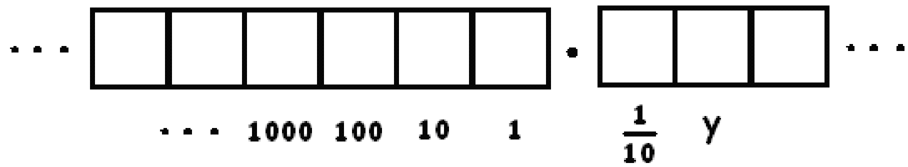
$$10x = 1$$

Ten times something equals one? We must have that x is the fraction $\frac{1}{10}$.



Call the value of the next box to the right of the decimal point *y*.

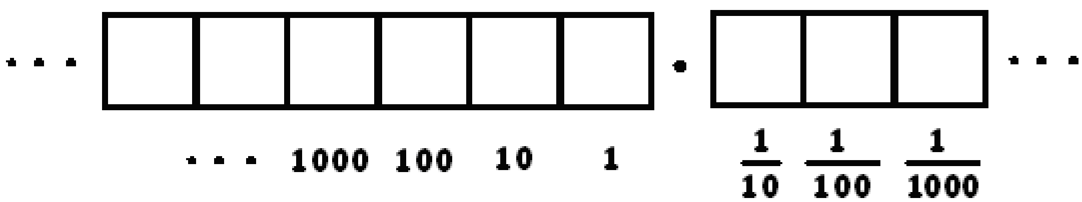
$1 \leftarrow 10$



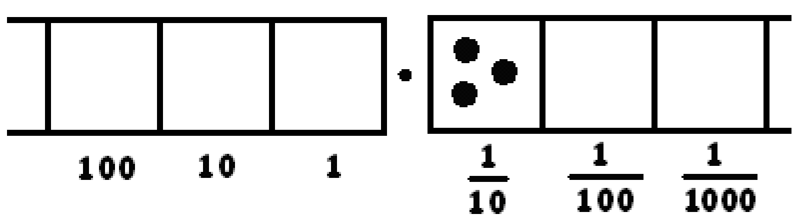
Since this is a $1 \leftarrow 10$ machine we must have $10y = \frac{1}{10}$. Thus *y* is the fraction $\frac{1}{100}$ (because $10 \times \frac{1}{100} = \frac{10}{100}$ which does indeed equal $\frac{1}{10}$).

If we keep doing this we see that the boxes to the right of the decimal point represents powers of ten as fractions.

$1 \leftarrow 10$



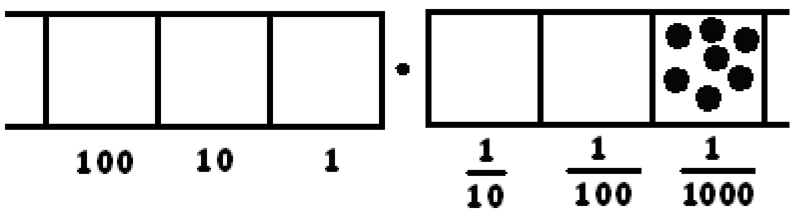
EXAMPLE: The decimal 0.3 is represented by the picture:



It represents three groups of $\frac{1}{10}$, that is:

$$0.3 = \frac{3}{10}$$

EXAMPLE: The decimal 0.007 is represented by the picture:



It represents the fraction $\frac{7}{1000}$

Question 1:

a) What fractions do the following decimals represent:

0.09 0.003 0.7 0.0000003

b) Write the following fractions as decimals:

$\frac{1}{1000}$ $\frac{7}{100}$ $\frac{9}{10}$

Of course, some decimals represent fractions that can simplify (reduce) further. For example:

$$0.5 = \frac{5}{10} = \frac{1}{2}$$

Conversely, if a fraction can be rewritten to have a denominator that is a power of ten, then it is easy to convert it to a decimal. For example, $\frac{3}{5}$ can be written as $\frac{6}{10}$ and so we have:

$$\frac{3}{5} = 0.6$$

Comment: Revisit explorations 10 and 11 if you need a reminder on working with fractions.

Question 2:

a) What fractions (in simplest terms) do the following decimals represent:

0.05 0.2 0.8 0.004

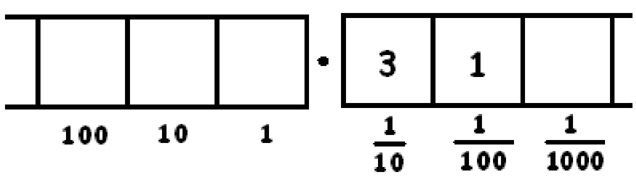
b) Write the following fractions as decimals:

$\frac{2}{5}$ $\frac{1}{25}$ $\frac{1}{20}$ $\frac{1}{200}$ $\frac{2}{2500}$

Question 3: Some people read 0.6, for example, out loud as "point six." Others read it out loud as "six tenths." Which is more helpful for understanding what the number really is? Why do you think so?

Here is a more interesting question:

What fraction is represented by the decimal 0.31?



There are two ways to think about this.

<u>Approach 1</u>: From the picture of the $1 \leftarrow 10$ machine we see:

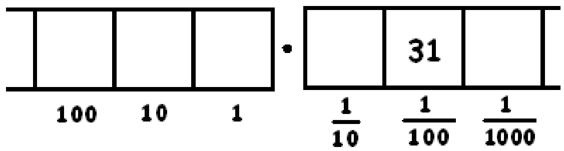
$$0.31 = \frac{3}{10} + \frac{1}{100}$$

We can add these fractions by find a common denominator:

$$\frac{3}{10} + \frac{1}{100} = \frac{30}{100} + \frac{1}{100} = \frac{31}{100}$$

Thus 0.31 is the fraction $\frac{31}{100}$.

<u>Approach 2</u>: Let's unexploded the three dots in the $\frac{1}{10}$ to produce an additional 30 dots in the $\frac{1}{100}$ position.

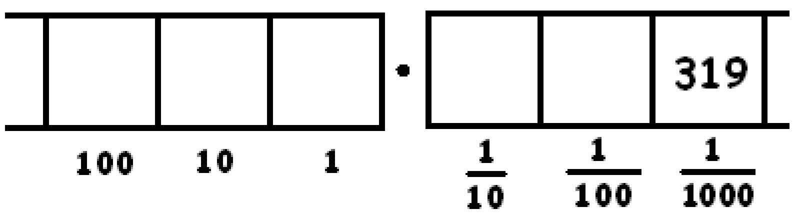


Thus we can see that $0.31 = \frac{31}{100}$.

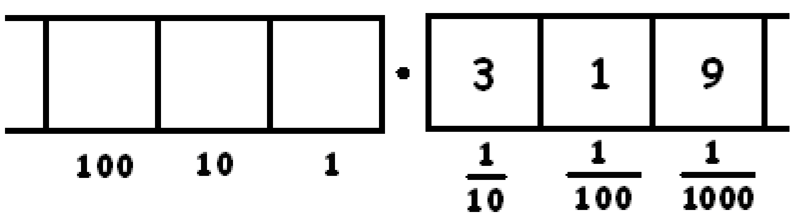
Question 4: Brian is having difficulty seeing that 0.47 represents the fraction $\frac{47}{100}$. Describe the two approaches you could use to help explain this to him.

Question 5: A teacher asked his students to each draw a $1 \leftarrow 10$ machine picture of the fraction $\frac{319}{1000}$.

JinJin drew:



Subra drew:



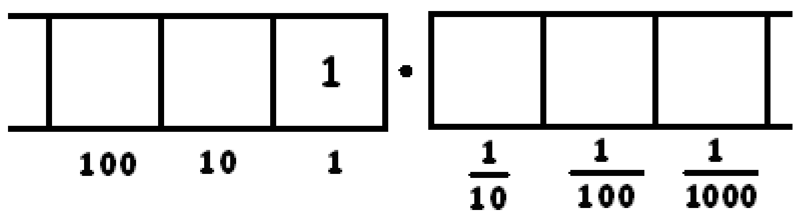
The teacher marked both students as correct. Are each of these solutions indeed valid? Explain your thinking.

JinJin said all that she would need to do in order to get Subra's solution is to perform some explosions. What did she mean by this? Is she right?

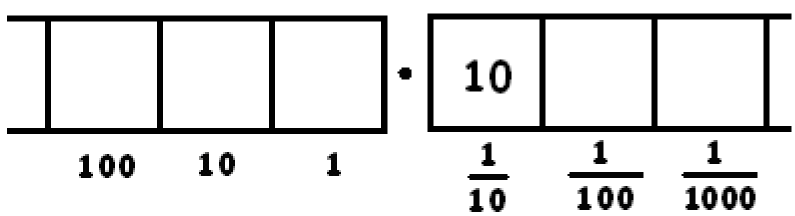
B. RETURNING TO DIVISION

Recall from exploration 10 that a fraction is just a division problem in disguise. For example, the fraction $\frac{1}{8}$ is the result of dividing one whole into eight parts.

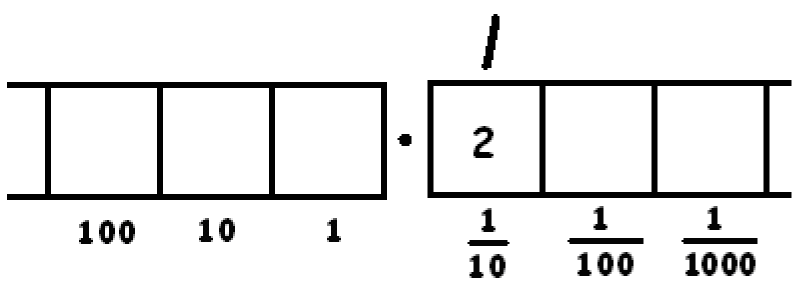
Let's actually perform this division problem in a $1 \leftarrow 10$ machine, making use of decimals. We seek groups of 8 in the following picture:



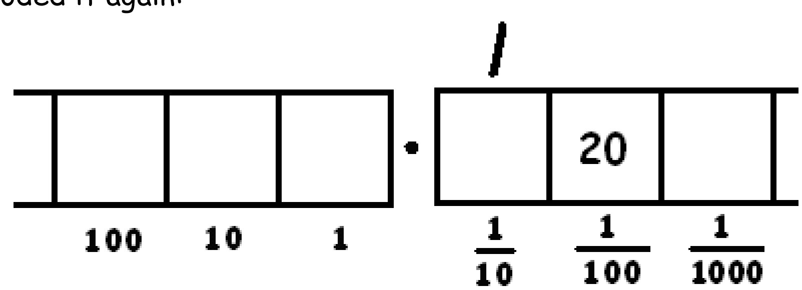
Clearly none are to be found, so let's unexplode:



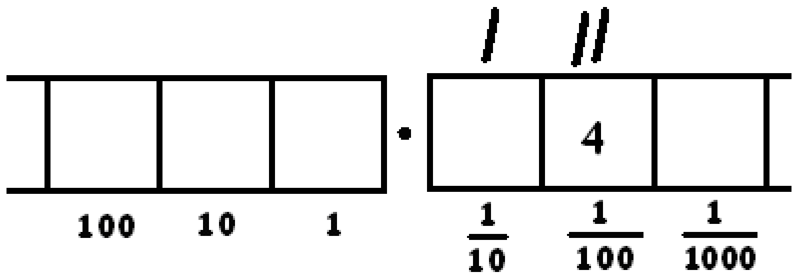
Now there is one group of 8, leaving two behind:



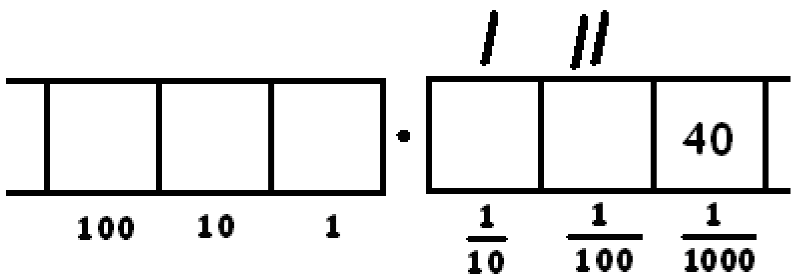
Let's unexploded it again:



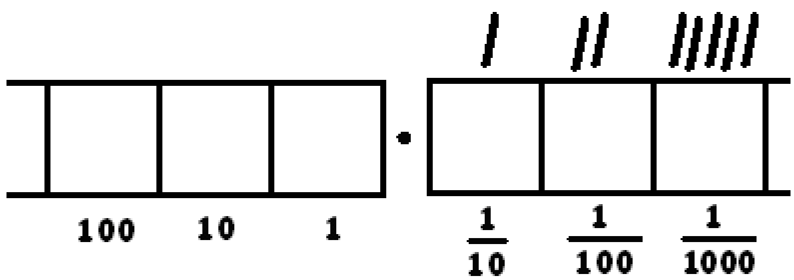
This gives two groups of 8 leaving four behind:



Unexploding again:



And here we have five groups of 8 leaving no remainders:



We now see that as a decimal, $\frac{1}{8}$ turns out to be 0.125.

Comment: And backwards … $0.125 = \frac{125}{1000} = \frac{1\times 125}{8\times 125} = \frac{1}{8}$

Question 11:

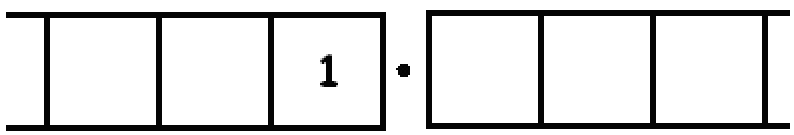
a) Perform the division in a $1 \leftarrow 10$ machine to show that $\frac{1}{4}$, as a decimal, is 0.25.

b) Perform the division in a $1 \leftarrow 10$ machine to show that $\frac{1}{2}$, as a decimal, is 0.5.

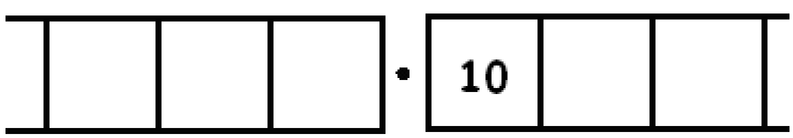
c) Perform the division in a $1 \leftarrow 10$ machine to show that $\frac{3}{5}$, as a decimal, is 0.6.

d) **CHALLENGE:** Perform the division in a $1 \leftarrow 10$ machine to show that $\frac{3}{16}$, as a decimal, is 0.1875.

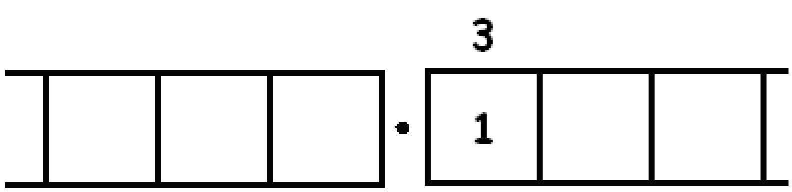
Not all fractions lead to simple decimal representations. For example, consider the fraction $\frac{1}{3}$. We seek groups of 3 in the following picture.



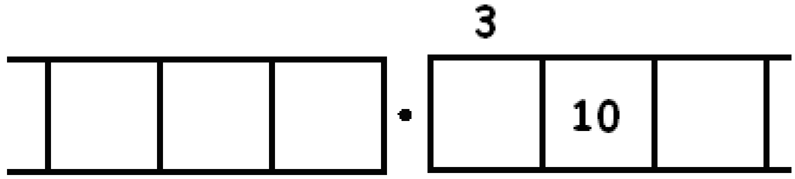
Unexploding requires us to look for groups of 3 instead in:



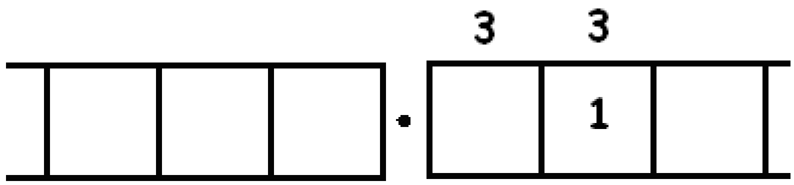
Here there are three groups of 3 leaving one behind:



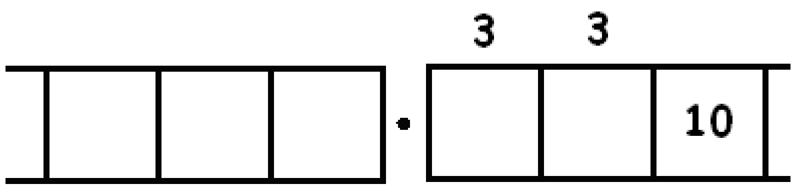
Unexploding gives:

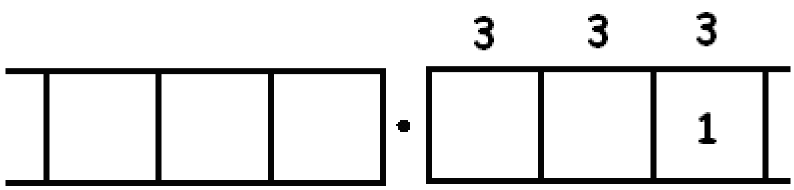


in which we find another three groups of 3 leaving one behind:



Unexploding gives:





and so on

and we seem to be caught in an infinitely repeating cycle.

We are now in a philosophically interesting position. As human beings, we cannot conduct this, or any, activity for an infinite amount of time. But it seems very tempting to write:

$$\frac{1}{3} = 0.33333\cdots$$

with the ellipsis "$\cdots$" representing the instruction "keep going forever with this pattern." In our minds it seems we can almost imagine what this means, but as a practical human being it is beyond our abilities: one cannot actually write down those infinitely many 3s represented by $\cdots$.

Nonetheless, many people choose not to contemplate what an infinite statement means and like to carry on and say: "Some decimals are infinitely long" and simply not be worried by it. In which case, the fraction $\frac{1}{3}$ is one of those fractions whose decimal expansion goes on forever!

COMMENT: Many people make use of a vinculum to represent infinitely long repeating decimals. For example, $0.\overline{3}$ means "repeat the 3 forever":

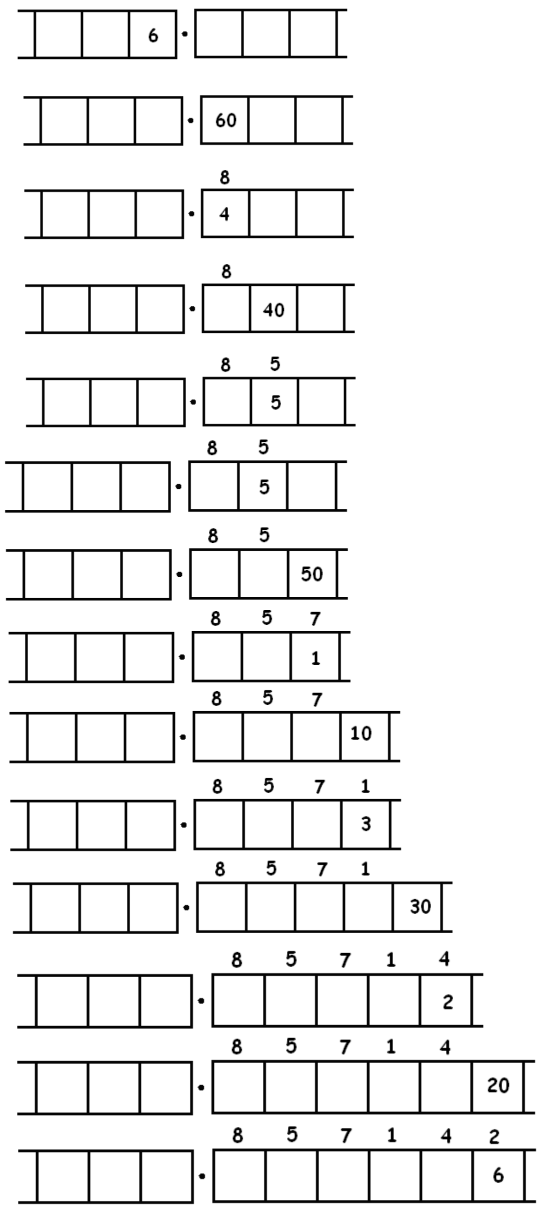
$$0.\overline{3} = 0.3333\cdots$$

and $0.\overline{142}$ means "repeat 142 forever":

$$0.\overline{142} = 0.142142142142\cdots$$

Notice that the vinculum is still being used in its correct sense to bring together symbols as a single group!

As another (complicated) example, here is the work that converts the fraction $\frac{6}{7}$ to an infinitely long repeating decimal. Make sure to understand the steps one line to the next.



Do you see, with this 6 in the final right-most box that we have returned to the very beginning of the problem? This means that we shall simply repeat the work we have done and obtain the same sequence 857142 of answers, and then again, and then again.

We have: $$\frac{6}{7} = 0.857142857142857142857142\cdots$$

Question 13:

a) Compute $\frac{4}{7}$ as an infinitely long repeating decimal.

b) Compute $\frac{1}{11}$ as an infinitely long repeating decimal.

Question 14: Which of the following fractions give infinitely long decimal expansions?

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \frac{1}{8} \quad \frac{1}{9} \quad \frac{1}{10}$$

We began this exploration with the problem of computing $1024 \div 3$. We found the answer 341 with one dot yet to be divided by three:

$$1024 \div 3 = 341 \ R \ 1$$

In terms of fractions this translates to:

$$1024 \div 3 = 341\frac{1}{3}$$

and in terms of decimals we have:

$$1024 \div 3 = 341.33333\cdots$$

Question 15:

a) Use a $1 \leftarrow 10$ machine to compute $133 \div 6$, writing the answer as a decimal.

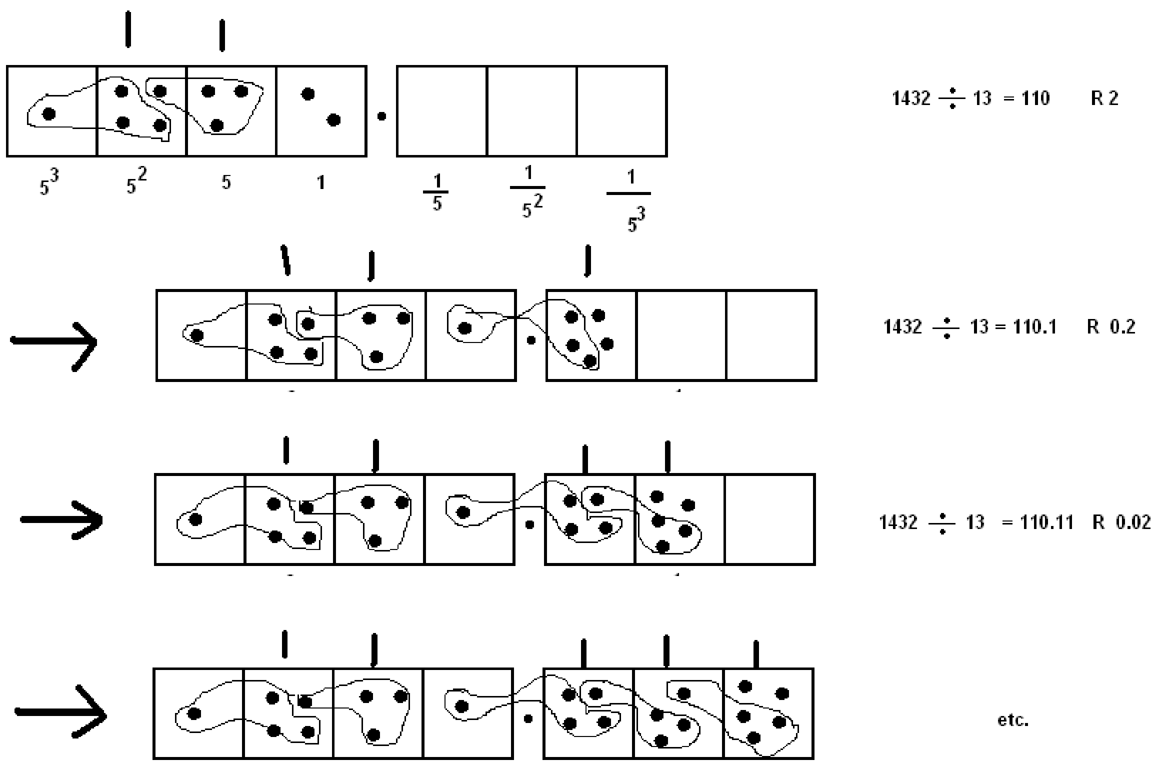
b) Use a $1 \leftarrow 10$ machine to compute $255 \div 11$, writing the answer as a decimal

D. EXPLORATION

The prefix *deci-* means "ten" (a decade is ten years and December used to be the tenth month of the year!) and our exploration of decimals in this chapter have all been with regard to a $1 \leftarrow 10$ machine and the power of ten. But we can repeat this work in any machine we desire.

For example, let's work with a $1 \leftarrow 5$ machine and the powers of five. (Thus, five dots in any one box "explode" to become one dot one place to the left.)

The following picture computes $1432 \div 13$ in a $1 \leftarrow 5$ machine:



We see that $1432 \div 13 = 110.1111...$ in base 5.

ADVENTURE 1:
Compute $1 \div 11$ in $1 \leftarrow 3$ machine and show that it yields the answer 0.020202….

Comment: The number "11" in a $1 \leftarrow 3$ machine is the number $1 \times 3 + 1 = 4$. This problem has just shown that the fraction $\frac{1}{4}$ in base three is 0.020202….

ADVENTURE 2: Compute $2 \div 5$ in $1 \leftarrow 4$ machine.

