Math Teachers' Circle
An introduction to problem solving

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In this set of problems, footnotes are for hints: don't read them right away!
We'll start with the "numbers 1 through 100," problem 17. The idea is that, instead of starting at the beginning and having me tell you what to do and what to think about, by doing this one problem you will discover for yourselves all the techniques I wanted to show, and then my role will become naming the techniques and giving additional examples of each.

The most important, for adults as well as for kids, is

Patience

because without it, you won't get very far. Especially for kids, who may be used to solving everything in their math class in a few seconds, they need to learn that real problems require real patience. A good way to illustrate this is with problems where some insight can speed things up, but there's still going to be some trial-and-error involved, such as:

1. On a 5x5 chessboard, place 5 wolves (who can move like chess queens) and 3 sheep so that all the sheep are safe from being eaten by the wolves.

   Generalize! What about w wolves and s sheep on an n by n board?

2. Triangle problem: arrange the numbers 1 through 6 into a "difference triangle", where each number in the row below is the difference of the two numbers above it. For example, 6 4 1, 2 3, 1 almost works, but it has two 1s and no 5. Variations: What about with 10 numbers? 15? What is the solution with the smallest range in the top row (largest minus smallest number)?

3. Find a 9-digit number using each digit 1 through 9 once, such that the first n digits are divisible by n.

OK, Patience alone is not enough: you have to do something while you're being patient! Pencils have to be moving! Math is not a spectator sport! Insert your favorite cliché here! Call it

Getting Dirty

Which is to say, you can't expect to fix your plumbing problem if you insist on your hands staying clean the whole time. Try **special cases**: doodle, try some examples, try some different (easier) versions of the problem, try some numbers, guess and check. Never start by staring at a blank piece of paper. Later, after you have some food for thought, pausing by staring at a non-blank piece of paper might be OK.

4. What's the smallest number that can't be written by subtracting a prime from a square?
   For example, 1 = 4 – 3, 2 = 9 – 7, …

5. The locker problem: begin with 1000 lockers, all open. Then 1000 people walk by, with the n^{th} person switching (from open to closed, or from closed to open) all the lockers whose numbers are divisible by n. Which lockers are closed after all 1000 people have walked by? [How would you change the statement of this problem to make it more understandable? **Restating** the problem is often a good strategy for starting the "getting dirty" process.]

6. Around the Round Table sit the 64 knights. Sir Cumference then walks around and around the table, saying alternately "go, stay, go, stay" for each knight he passes. As they obey and go, of course, Sir Cumference doesn't talk to the empty chairs. Which knight will be the last one remaining? [Don't forget to start with special cases, and end by generalizing!]

Once you're getting dirty, you can produce a lot of ideas. And if you're patient, too, you can really produce a lot of scratch work. So, the next key skill,

Organization

7. I'm sure many of you know one or more clever tricks for adding up the numbers 1 through 100, but have you ever tried just adding them? What patterns are there that you can exploit? Is there a way you can organize it?[1]

8. Instead of the numbers 1 through 100, add up all the numbers in the usual multiplication table. What are different ways to find that sum?[2]

9. How many rectangles are there in the 10x10 chessboard?[3]

10. How many factors does 10 have? 20? 30? 36? 120? 496? n? There are lots of different ways to describe this – share!

11. How many ways are there to make change for a dollar?[4]

12. Arrange the numbers 1 through n in a circle, such that the highest LCM of adjacent numbers is as small as possible. For instance, with 1,2,3,4 you could arrange them in that order and have highest LCM 12. But if you arrange them as

[1] Use groups of 10, perhaps.

[2] Can you add up each row? Have you ever used lattice multiplication? How about summing antidiagonals — can you find it in Pascal's triangle?

[3] Is there a reason this comes right after the previous problem? Can you count how many rectangles have a given spot as their bottom right corner? Where can the top left corner be?

[4] What if there were only pennies? Only pennies and nickels? Pennies, nickels, and dimes? Special cases, trying a simpler problem, and getting dirty. Recursion, using answers to simpler problems to produce an answer to a bigger problem, is also very useful at times.

1,3,2,4 instead, then the highest LCM is 6. So for $n = 4$, the answer is 6.

How can you be sure you found the best? Do you have to try every possibility?

What variations of this problem can you think of?

Visit the online encyclopedia of integer sequences and look up your special cases to see if a pattern is known! http://www.research.att.com/~njas/sequences

You've been doing it all along, as a way of pushing to the next step: once you solve a problem, think of variations, and most especially see if you can

Generalize

The most amazingly surprising thing is that generalizing sometimes makes a problem easier. You can prove something more with less work! It's not always true that variables are scarier or harder! Once you have generalized, among other things, you can use special cases to help. For example:

13. The semimajor axis of an ellipse is 3 inches long and its semiminor axis is 2 inches long. What is its area?
    (A) 13π sq in
    (B) $13\pi/2$ sq in
    (C) 6π sq in
    (D) 5π sq in
    (E) $5\pi/2$ sq in

Now, you might know the formula, but if you don't, it's probably hard to choose among some of those answers. But when the same answers are given with variables:

14. The semimajor axis of an ellipse is a and its semiminor axis is b. What is its area?
    (A) $\pi (a^2 + b^2)$
    (B) $\pi (a^2 + b^2)/2$
    (C) πab
    (D) $\pi (a^2 - b^2)$ [or maybe $\pi (a + b)$]
    (E) $\pi (a^2 - b^2)/2$ [or maybe $\pi (a + b) / 2$]

Having generalized, special cases saves the day: think of the circle, where $a = b$ and you know the formula. Or, think of the squashed ellipse where b gets very close to 0 and so does the area. There you have it! [For more on this sort of reasoning, see Barry Cipra's excellent little book, *Misteaks: and how to find them before the teacher does*, which is mostly aimed at calculus students]

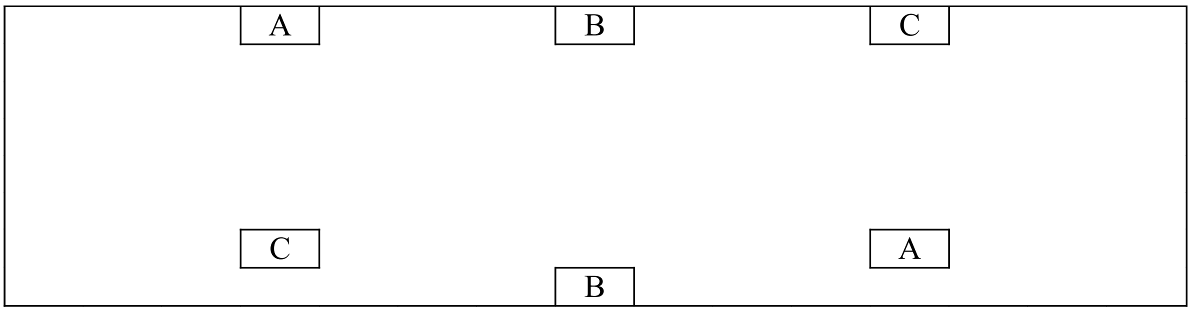
So, simplifying problems is a good thing. You can do it with special cases, and sometimes you can even do it by generalizing. Another good way is to use a little

Wishful Thinking

My favorite example:

15. Factor $x^4 + x^2 + 1$

16. Another one from Paul Zeitz: connect A to A, B to B, and C to C, without crossing lines or leaving the box.



17. The numbers 1 through 100 are written on the board. Each minute, you choose any two numbers u and v and erase them, writing $uv + u + v$ in their place. After 99 minutes, only one number will be left on the board. What possible numbers might that be? You can combine a lot of the above techniques in solving this problem, a minor variation of one I heard from Paul Zeitz.

Another technique that probably came up in the exploration of this last problem is

Symmetry

18. Solve
    $Ax + By = C$
    $Dx + Ey = F$.[5]

19. Compute cos 7° + cos 79° + cos 151° + cos 223° + cos 295°. No calculators![6]

20. Try to guess a formula for the frustum of a cone (that is, a piece of a cone with its tip cut off, with base radius R and upper radius r and height h). How can you use symmetry (and special cases, and perhaps other techniques) to help you? Of course you can also find the formula using algebra.

And a few more assorted problems just in case we need some more to keep us busy, again some of which are stolen from Paul Zeitz, and mostly on the theme of

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[5] Solving for x should be done the usual way, but then solving for y can be almost instant.
[6] A good technique that I didn't mention before, but probably should have: **make a picture**.

Invariants

which are those things that stay the same when other things change. For example, if you're only ever adding 2, then the oddness or evenness of your starting number will always stay the same.

21. A mad veterinarian (who can be found online at http://www.bumblebeagle.org/madvet/index.html along with many other puzzles) has three animal transmogrifying machines. The first turns a cat into two dogs and five mice. The second turns a dog into three cats and three mice. And the third turns a mouse into a cat and a dog. Each machine can also operate in reverse. If you start with three cats, what's the smallest number of mice you can end up with? What's the fewest number of cats that can be turned into pure dogs? Invent your own questions, or visit the web site to find other interesting combinations of machines of various levels of difficulty.

22. A cube begins with one vertex labeled 1 and all the rest labeled 0. At any moment, you can add 1 to each of any two adjacent vertices. Can you ever make the vertices all have multiples of 3?

23. Three numbers are written on the board. At any moment, you can erase one of the numbers and replace it with the sum of the other two, minus 1. If you end up with 17 1967 1983, what numbers might you have started with? Could it have been 2 2 2? 3 3 3?

24. You start with the point (7, 29). If you have the point (x,y) you can add $(x+1, y+1)$ to your collection. If x and y are both even, you can make $(x/2, y/2)$ also. And, if you also have (y, z) you can make (x,z). Can you ever make (7, 31)? (7,1999)? (7,2006)? (0,0)? What points can you make?

For middle school, some of these problems are obviously nonsense, and others are just way too hard. Problem 1 through 11 should be good, though most of them would take most of a class period to do. Problems 16 and of course 17 are great. Problem 21 is a lot of fun, and you can make easier versions of it, like this:

25. A mad veterinarian has three animal transmogrifying machines.
    Machine 1 turns two cats into one cat.
    Machine 2 turns a cat and a dog into one dog.
    Machine 3 turns two dogs into one cat.
    Each machine can also operate in reverse.
    Can the veterinarian begin with three cats and one dog, and end with just one cat?

    What starting situations can end with just one cat?

Problem 22, 23, and 24 are a bit on the tough side.

More stuff for the LCM organization problem:
Answer: once you've found a good one, you can eliminate any adjacencies that give a bigger LCM than your established best, which greatly cuts down on the possibilities to check.

Is there any systematic way to work this out for large n?

Vary the problem, and generalize! Sometimes you'll find easier problems, sometimes harder, sometimes more interesting.

If you do it with products instead, then:
For odd $n > 2$, $a(n) = (n+1)(n+3)/4$. For even $n > 2$, $a(n) = n(n+4)/4$.

For products in a row instead of a circle:
Theorem: $a(n)=n*(n+2)/4$ if n is even and $(n-1)*(n+3)/4$ if n is odd,
$a(n) = a(n-1)+a(n-2)-a(n-3)+1$ = A002620(n+2)+A004526(n+2)

What about sums? Absolute differences?