☆ EXPLODING DOTS ☆

ADVANCED VERSION FOR PERSONAL FUN

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This material is adapted from the text:

THINKING MATHEMATICS!
Volume 1: Arithmetic = Gateway to All

available at the above website.

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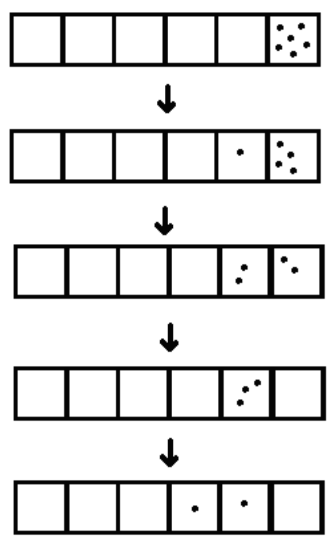
BASE MACHINES

Here's a simple device that helps explain, among other things, representations of numbers.

A $1 \leftarrow 2$ base machine consists of a row of boxes, extending as far to the left as one pleases. To operate this machine one places a number of dots in the right most box, which the machine then redistributes according to the following rule:

> *Two dots in any one box are erased (they "explode") to be replaced with one dot one box to their left.*

For instance, placing six dots into a $1 \leftarrow 2$ machine yields four explosions with a final distribution of dots that can be read as "1 1 0."

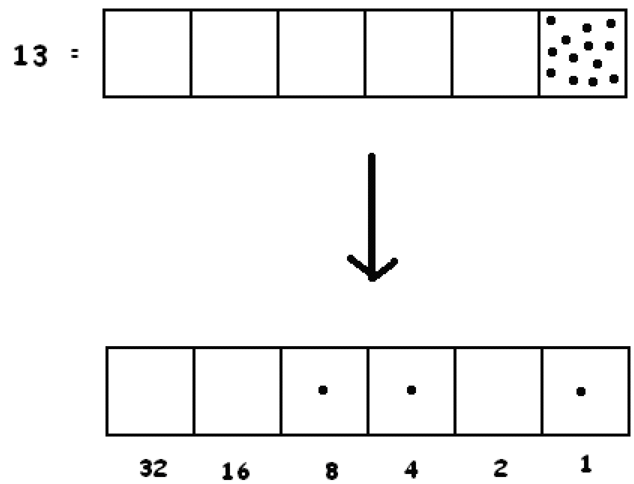


Placing 13 dots into the machine yields the distribution "1 1 0 1" and placing 50 dots into the machine, the distribution "1 1 0 0 1 0."

EXERCISE: Check these.

EXERCISE: Does the order in which one chooses to conduct the explosions seem to affect the final distribution of dots one obtains?
(This is a deep question!)

Of course, since one dot in a cell is deemed equivalent to two dots in the preceding cell, each cell is "worth" double the cell to its right. If we deem the right-most cell as the units, then each cell of the machine corresponds to the powers of two. Thirteen, for instance, equals 8 + 4 + 1 and the base-2 representation of 13 is 1101.



The $1 \leftarrow 2$ machine converts all numbers to their binary representations.

In the same way, a $1 \leftarrow 3$ machine (three dots explode to make one dot to the left) gives the base-3 three representations of a number and a $1 \leftarrow 10$ machine the base-10 representations.

EXERCISE:

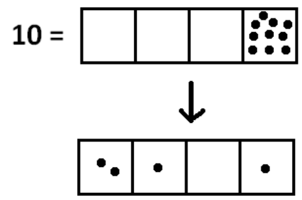
a) Use a $1 \leftarrow 3$ machine to find the base-3 representations of the numbers 8, 19 and 42.

b) Use a $1 \leftarrow 5$ machine to find the base-5 representation of the number 90.

c) Use a $1 \leftarrow 10$ machine to find the base-10 representation of the number 179. (Actually think your way through this!)

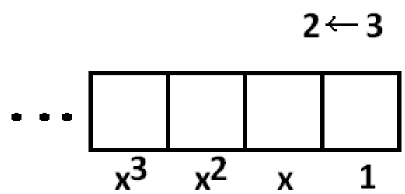
<u>COMMENT</u>: You might find it easiest to conduct this work on a chalk board. Erasing dots on paper is not easy.

One can have fun with base machines and develop some interesting mathematical curiosities. Consider, for example, a $2 \leftarrow 3$ machine. This machine replaces three dots in one box with two dots one place to the left. Placing ten dots in the machine produces the following result:



$$10 \to 2101$$

(Check this!) What base is this machine?



We have that three units is "worth" two xs ($2x = 3 \cdot 1$) and three xs is worth two x^2s ($2x^2 = 3x$), and so on. This gives $x = \frac{3}{2}$. We have a base one-and-a-half machine! And as a check, it is true that $2 \times \left(\frac{3}{2}\right)^3 + 1 \times \left(\frac{3}{2}\right)^2 + 0 \times \frac{3}{2} + 1 \times 1$ equals ten! (Check this.)

OPTIONAL EXERCISE: Write the number 1 through 30 in base one-and-a-half using a $2 \leftarrow 3$ machine. Any patterns?

OPTIONAL EXERCISE: In what way a $2 \leftarrow 4$ machine differ from a $1 \leftarrow 2$ machine? (They should both be some form of base-2 representations.)

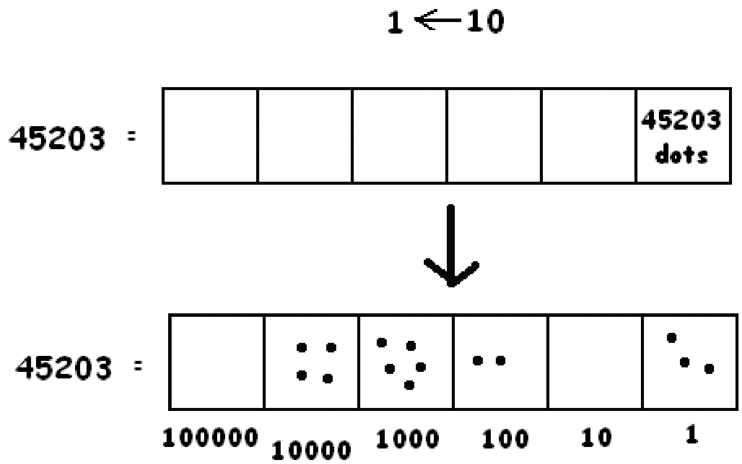
Write the numbers 1 through 30 as given by a $2 \leftarrow 4$ machine and as given by a $1 \leftarrow 2$ machine. Does there seem to be an easy way to convert from one representation of a number to the other?

OPTIONAL EXERCISE: Discuss the utility of a $1 \leftarrow 1$ machine. Discuss the utility of a $2 \leftarrow 1$ machine.

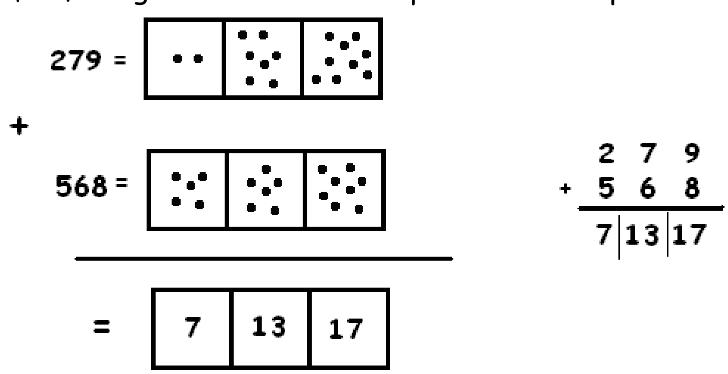
MORE ON THESE AND OTHER CRAZY MACHINES IN THE EXPLORATIONS AT THE END OF THIS PACKET (Page 21)

ARITHMETIC IN A $1 \leftarrow 10$ MACHINE

The positional notational system we use today for writing and manipulating numbers is precisely the $1 \leftarrow 10$ machine in action.



ADDITION: Addition in a $1 \leftarrow 10$ machine is straightforward. For instance, adding together 279 and 568 yields 2 + 5 = 7 dots in the 100s position, 7 + 6 = 13 dots in the 10s position, and 9 + 8 = 17 dots in the 1s position. It is reasonable to write this answer as 7 | 13 | 17, using vertical bars to separate unit of powers of 10.



Since ten of the 13 dots in the 10s cell "explode" to leave three behind and create one extra dot in the 100s place our answer is equivalent to 8 | 3 | 17. Also, for the 17 dots in the 1s place, ten can explode to leave 7 behind and to create an extra dot in the 10s place. We have the final answer $279 + 568 = 8 | 4 | 7$, which is just the number 847. (The process of exploding tens of dots is usually called *carrying*.)

It is actually much quicker and easier to complete addition problems column-wise from left to right (rather than right to left as is traditionally taught) and leave "carrying" until the end of the process.

EXERCISE:

a) Quickly solve the following addition problems by working left to right and leaving all carrying of digits to last.

$$\begin{array}{r} 76521 \\ +14774 \\ \hline \end{array} \qquad \begin{array}{r} 5732294 \\ +\ 2394826 \\ \hline \end{array}$$

b) Compute 56243×7 by leaving all carrying to the end.

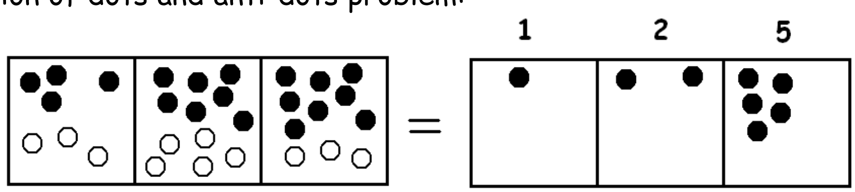
SUBTRACTION:

Subtraction is nothing but the addition of negative quantities. In our dots and boxes model let's work now with dots and "anti-dots." We will represent dots as solid circles and shall represent anti-dots as hollow circles and note that each anti-dot annihilates an actual dot.

The problem:

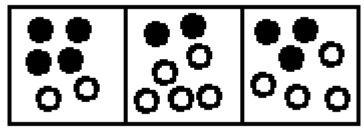
$$\begin{array}{r} 4\ 7\ 8 \\ -\ 3\ 5\ 3 \end{array}$$

is an addition of dots and anti-dots problem:

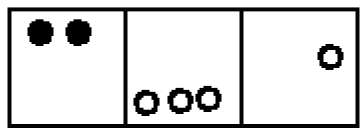


$$\begin{array}{r} 4\ 7\ 8 \\ -\ 3\ 5\ 3 \\ \hline =\ 1\ 2\ 5 \end{array}$$

As another example we see that 423 – 254, represented by diagram:



has answer:



That is:

423 – 254 = 2|-3|-1

This is absolutely valid mathematically, though the rest of the world may have difficulty understanding what "two hundred and negative thirty negative one" means! To translate this into familiar terms we can "unexplode" one of the solid dots to add ten solid dots to the middle box. This gives the representation: 1 | 7 | -1. Unexploding again gives: 1|6|9.

Thus we have:

423 – 254 = 169.

<u>COMMENT</u>: In grade-school this process of "unexploding" is called "borrowing digits." Young students are taught to unexplode as they work though a subtraction problem rather than leave all the unexploding until the end.

EXERCISE: Use this method, working from left to right (and not right to left as one is taught in grade school) to compute the following quantities:

```
   6 3 2 8          7 8 3 9 0 2 3 1
 - 4 4 6 9        - 3 2 4 9 5 8 4 6
 ---------        -----------------
```

MULTIPLICATION:
Some simple multiplication problems are conceptually clear in the dots and boxes model. For example:

$$\begin{aligned}45076\times 3 &= 12\,|\,15\,|\,0\,|\,21\,|\,18\\ &= 13\,|\,5\,|\,0\,|\,21\,|\,18\\ &= 13\,|\,5\,|\,2\,|\,1\,|\,18\\ &= 13\,|\,5\,|\,2\,|\,2\,|\,8 \quad = \quad 135228.\end{aligned}$$

DIFFICULT CHALLENGE: Develop a method for performing long multiplication in the dots-and-boxes model.
Use your method to compute 354×672.

Although it is possible to develop long-multiplication procedures in the dot-and-boxes model, they tend to be somewhat artificial. This is a limitation of this model.

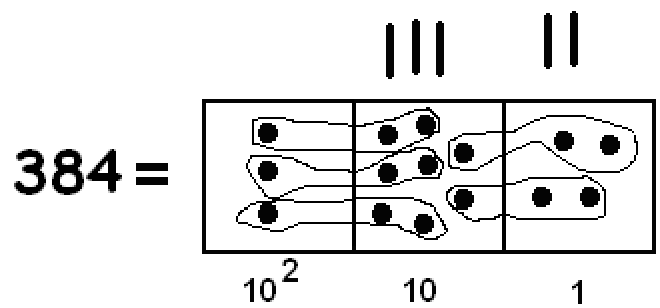
LONG DIVISION
Despite its limitation with multiplication, one significant advantage of the dots-and-boxes model is that it explains the process of long division with conceptual ease.

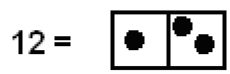
Consider, for instance, the problem of dividing 384 by 12. This means we wish to count the number of times we can find groups of 12 within 384 dots. Here's the traditional algorithm for doing this:

```
      32
    ____
12 ) 384
     36↓
    ----
      24
      24
    ----
       0
```

This algorithm is quite mysterious and devoid of straightforward meaning.

As a dots-and-boxes problem, we are seeking groups of 12 within 384. And this is just a task of recognizing patterns:





Notice that we recognize three group of 12 at the tens level (namely, one hundred and two tens) and two groups of 12 at the units level (namely, one ten and two units).

Thus $384 \div 12$ is indeed 32.

COMMENT: Can you see that the traditional long-division algorithm actually is this process in disguise?

EXERCISE: Compute the following division problems using the dots-and-boxes method:

a) $235431 \div 101$
b) $30176 \div 23$

COMMENT: One may have to "unexplode" dots along the way.

EXERCISE: Compute $2798 \div 12$ in a $1 \leftarrow 10$ machine and show that this problem leaves a remainder of 2.

COMMENT: The dots and boxes method is, as mentioned before, difficult to carry out on paper. (This method is not being advocated as a preferred method.) But do note, nonetheless, the conceptual ease of understanding that comes from the dots and boxes approach.

EXERCISE:
a) Use a $1 \leftarrow 5$ machine to show that, in base 5:

$$2014 \div 11$$

equals 133 with a remainder of one.

b) What is $2014 \div 11 = 133\ R\ 1$ as a statement in base 10? (That is, translate each of the numbers 2014, 11, 133, and 1 into their base 10 versions. Verify that the division statement is correct.)

EXERCISE: Quickly compute each of the following:

a) 263207×3
b) 563872×9
c) 673600023×2

d) Use the $1 \leftarrow 10$ machine to explain why multiplying a number in base 10 by 10 results in simply placing a zero at the end of the number.

e) Comment on the effect of multiplying a number written in base *b* by *b*.

EXERCISE: Here's a trick for multiplying two-digit numbers by 11:

To compute 14×11, say, split the 1 and the 4 and write their sum,5, in between:

$$14 \times 11 = 154$$

To compute 71×11 split the 7 and the 1 and write their sum, 8, in between:

$$71 \times 11 = 781$$

In the same way:

$$20 \times 11 = 220$$
$$13 \times 11 = 143$$
$$44 \times 11 = 484$$

It also works if one does not carry digits:

$$67 \times 11 = 6|13|\,7 = 737$$
$$48 \times 11 = 4\,|\,12\,|\,8 = 528$$

a) Why does this trick work?
b) Quickly, what's $693 \div 11$?
c) Quickly work out 133331×11.

A number is a palindrome if it reads the same way forwards as it does backwards. For example, 124454421 is a palindrome.

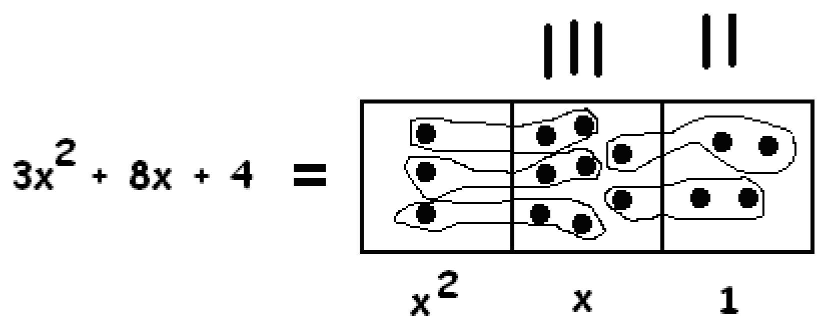
d) TRUE OR FALSE and WHY: Multiplying a palindrome by 11 produces another palindrome.

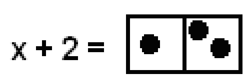
ARITHMETIC IN AN $1 \leftarrow x$ MACHINE

Much of the arithmetic we have conducted in a $1 \leftarrow 10$ machine is not bound to the specifics of base 10. We can, for instance, perform long division in a $1 \leftarrow 5$ machine or a $1 \leftarrow 8$ machine in exactly the same way.

In fact, if we simply write the base number of the machine as x, and work with an $1 \leftarrow x$ machine, then the same computation provides a method for dividing polynomials. For example, consider:

$$(3x^2 + 8x + 4) \div (x + 2)$$





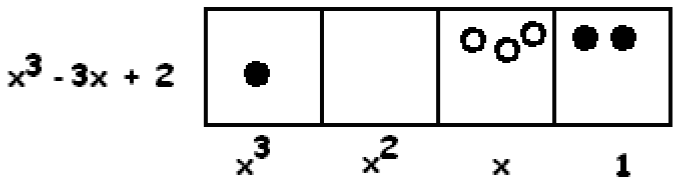
We see that $(3x^2 + 8x + 4) \div (x + 2) = 3x + 2$.

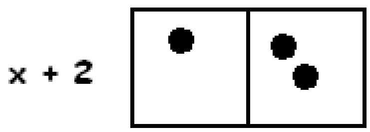
Thus the division of polynomials can be regarded as a computation of long division!

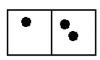
EXERCISE: Compute $\left(x^4 + 2x^3 + 4x^2 + 6x + 3\right) \div \left(x^2 + 3\right)$ using an $1 \leftarrow x$ machine.

Let's examine an example with negative coefficients. Let's compute $\left(x^3-3x+2\right)\div\left(x+2\right)$ using dots and anti-dots.

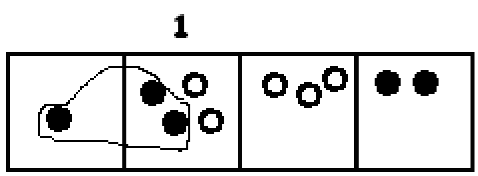
To do this, begin by drawing the representations of each polynomial.



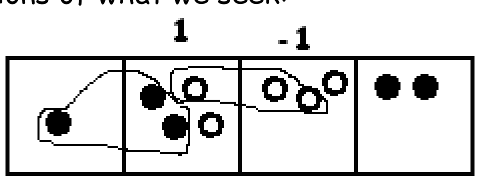


Our task is to find groups of  within the top diagram, and right away matters seem problematic. One might think to "unexplode" dots to introduce new dots (or anti-dots) into the diagram but there is a problem with this: We do not know the value of x and therefore do not know the number of dots to draw for each "unexplosion."

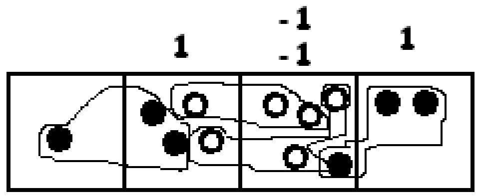
The way to cope with this difficulty is to employ an alternative trick: We can fill empty cells with dots and anti-dots. This keeps the value of the cell zero, but creates the dots we seek in our patterns.



It also creates anti-versions of what we seek:



Add another dot/antidot pair:



and we have:

$$\frac{x^3 - 3x + 2}{x + 2} = x^2 - 2x + 1.$$

(Check this by multiplying $x^2 - 2x + 1$ by $x + 2$.)

EXERCISE: Use dots-and-boxes to compute the following:

a) $\dfrac{x^3 - 3x^2 + 3x - 1}{x - 1}$

b) $\dfrac{4x^3 - 14x^2 + 14x - 3}{2x - 3}$

c) $\dfrac{4x^5 - 2x^4 + 7x^3 - 4x^2 + 6x - 1}{x^2 - x + 1}$

d) $\dfrac{x^{10} - 1}{x^2 - 1}$

Challenge: Is there a way to conduct the dots and boxes approach with ease on paper? Rather than draw boxes and dots, can one work with tables of numbers that keep track of coefficients? (The word “synthetic” is often used for algorithms one creates that are a step or two removed from that actual process at hand.)

EXERCISE: Use an $1 \leftarrow x$ machine to compute each of the following:

a) $\dfrac{x^2 - 1}{x - 1}$

b) $\dfrac{x^4 - 1}{x - 1}$

c) $\dfrac{x^6 - 1}{x - 1}$

d) Will $x^{\text{even}} - 1$ always be a multiple of $x - 1$?

e) Compute $\dfrac{x^6 - 1}{x + 1}$

f) Will $x^{\text{even}} - 1$ is also always be a multiple of $x + 1$?

g) Explain why $2^{100} - 1$ must be a multiple of 3 and be a multiple of 5.
(HINT: Let $x = 2^2$. Then $2^{100} - 1 = x^{50} - 1$.) Show that it is also a multiple of 33 and of 1023.

h) Is $x^7 + 1$ divisible by $x - 1$? Is it divisible by $x + 1$?

i) Is $x^{\text{odd}} + 1$ a multiple of $x - 1$? Of $x + 1$?

j) Explain why $2^{100} + 1$ is a multiple of 17. Show that $3^{100} + 1$ is a multiple of 41.

EXERCISE:

a) Compute $(x^6 + x^5 + 5x^4 + 5x^3 + 9x^2 + 5x + 2) \div (x^2 + x + 2)$.

b) Put $x = 10$ into your answer for a). What long division arithmetic problem have you solved?

EXERCISE: REMAINDERS

a) Using an $1 \leftarrow x$ machine show that $\frac{4x^4 - 7x^3 + 9x^2 - 3x - 1}{x^2 - x + 1}$ equals $4x^2 - 3x + 3$ with a remainder of $2x - 3$ yet to be divided by $x^2 - x + 1$.

(This means: $\frac{4x^4 - 7x^3 + 9x^2 - 3x - 1}{x^2 - x + 1} = 4x^2 - 3x + 2 + \frac{2x - 3}{x^2 - x + 1}$.)

b) Compute $\frac{x^4}{x^2 - 3}$

c) Compute $\frac{5x^5 - 2x^4 + x^3 - x^2 + 7}{x^3 - 4x + 1}$

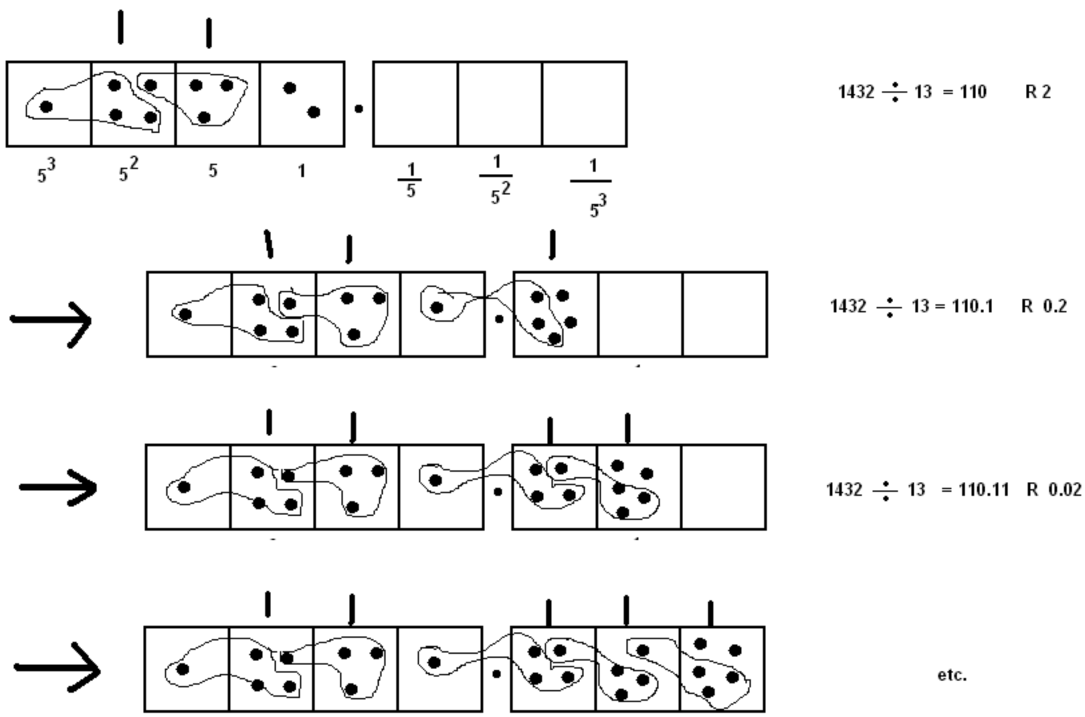
<u>HINT</u>: Drawing dots and anti-dots in cells is tiresome. Instead of drawing 84 dots (as you will need to do at one point for problem c) it is easier just to write "84."

AN INFINITE PROCESS: DECIMALS

As we know, the process of long division can produce nonzero remainders.

a) Show that, in base 5, dividing 1432 by 13 yields the answer 110 with a remainder of 2.

If one is willing to work with negative powers of five and keep "unexploding" dots, we can continue the long division process to see that, in base 5, $1432 \div 13 = 110.1111\ldots$.



b) Compute $8 \div 3$ in a base 10 machine and show that it yields the answer 2.666…

c) Compute $1 \div 11$ in base 3 and show that it yields the answer 0.020202….

(In base three, "11" is the number four, and so this question establishes that the fraction $\frac{1}{4}$ written in base 3 is 0.02020202…)

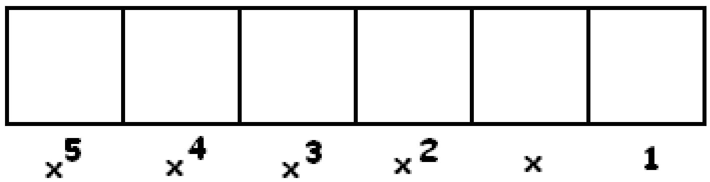
d) Show that the fraction $\frac{2}{5}$ (here written in base 10) has, in base 4, "decimal" representation $0.121212...$.

e) What fraction has decimal expansion 0.32323232… in base 7? Is it possible to answer this question by calling this number x and multiplying both sides by 10 and 100? (Does "100" represent one hundred?)

f) Written in base 9, let $x = 0.13131313...$. With numerator and denominator written in base 10, what fraction is x?

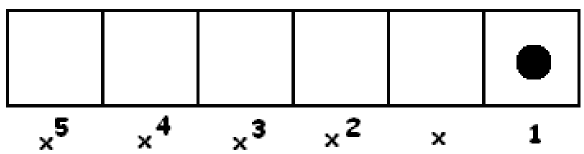
ANOTHER INFINITE PROCESS: THE GEOMETRIC SERIES

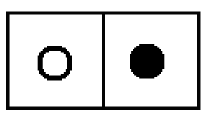
Consider again an $1 \leftarrow x$ base machine.

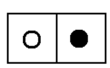
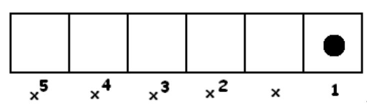


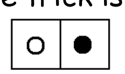
We can use this machine to divide 1 by $1-x$, that is, to compute $\frac{1}{1-x}$.

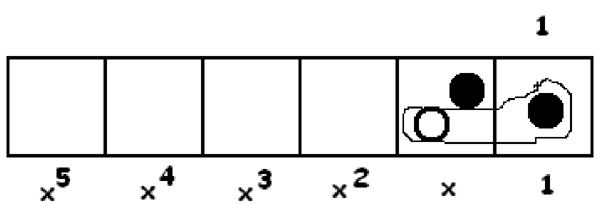
The quantity "1" is a single dot in the units position and the quantity "$1-x$" is an anti-dot in the x position.

1 = 

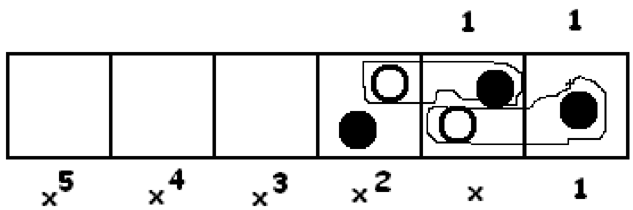
1-x = 

We wish to find copies of  in the picture  . Of course there are none at this stage.

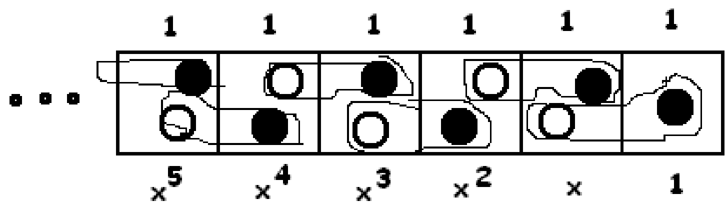
The trick is to fill and empty box with a dot and anti-dot pair. This gives us a copy of  in the units position.

1 = 

We can repeat this trick:



and again, infinitely often!



This shows that, as a statement of algebra, we have:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

a) Use this technique to show that $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 -$

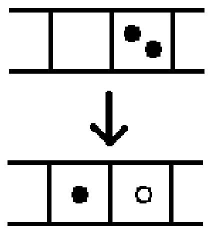
b) Compute $\frac{x}{1-x^2}$

c) Compute $\frac{1}{1-x-x^2}$ and discover the Fibonacci numbers!

EXPLORATIONS WITH CRAZY MACHINES

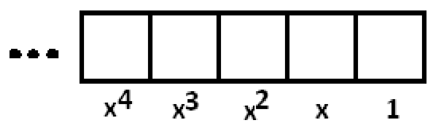
CRAZY MACHINE ONE: A DIFFERENT BASE 3

Here's a new type of base machine. It is called a $1 \mid -1 \leftarrow 0 \mid 2$ machine and operates by converting any two dots in one box into an anti-dot in that box and a proper dot one place to the left.



This machine also converts two antidots in one box to an antidot/dot pair.

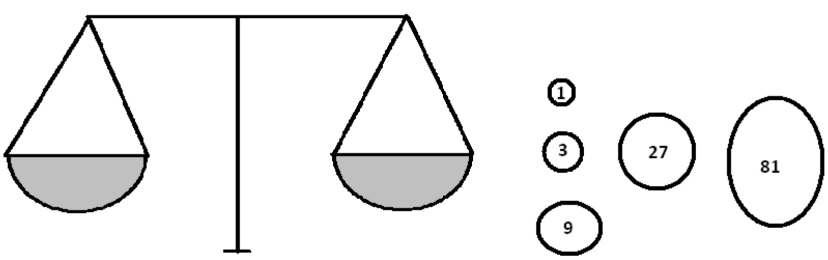
a) Show that the number 20 has representation $1 \mid -1 \mid 1 \mid -1$ in this machine.
b) What number has representation $1 \mid 1 \mid 0 \mid -1$ in this machine?
c) This machine is a base machine:



Explain why x equals 3.

Thus the $1 \mid -1 \leftarrow 0 \mid 2$ shows that every number can be written as a combination of powers of three using the coefficients 1, 0 and -1.

d) A women has a simple balance scale and five stones of weights 1, 3, 9, 27 and 81 pounds.

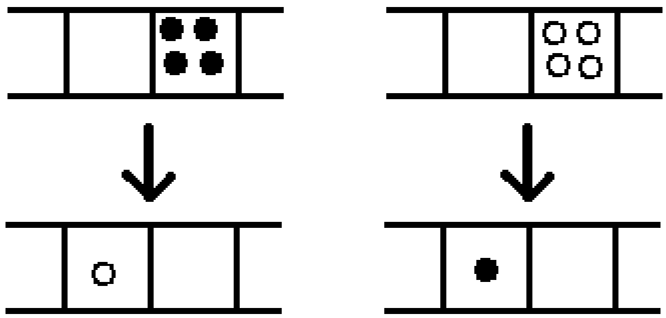


I place a rock of weight 20 pounds on one side of the scale. Explain how the women can place some, or all, of her stones on the scale so as to make it balance.

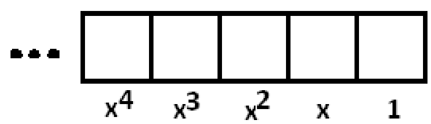
e) Suppose instead I place a 67 pound rock on the woman's scale?

CRAZY MACHINE TWO: BASE NEGATIVE FOUR

A $-1 \leftarrow 4$ machine and operates by converting any four dots in one box into an anti-dot one place to the left (and converts four antidots in one box to an actual dot one place to the left).



a) This machine is a base machine:



Explain why x equals -4.

b) What is the representation of the number one hundred in this machine? What is the representation of the number negative one hundred in this machine?

c) Verify that $2\,|-3\,|-1\,|\,2$ is a representation of some number in this machine. Which number? Write down another representation for this same number.

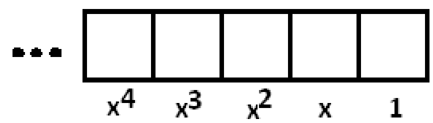
d) Write the fraction $\frac{1}{3}$ as a "decimal" in base -4 by performing long division in a $-1 \leftarrow 4$ machine. Is your answer the only way to represent $\frac{1}{3}$ in this base?

CRAZY MACHINE THREE: BASE TWO, BASE TWO, BASE TWO

We have seen at the start of these notes that a $1 \leftarrow 2$ machines converts numbers into their base-two representations.

The exercise on page 4 asks us to consider a $2 \leftarrow 4$ machine. In case you haven't yet done it …

a) Verify that a $2 \leftarrow 4$ machine is indeed a base two machine.

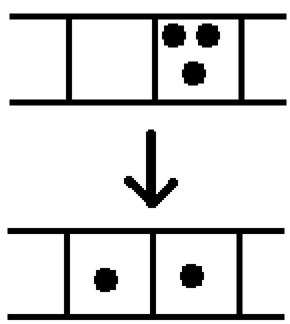


That is, explain why $x = 2$ is the appropriate value for x in this machine.

b) Write the numbers 1 through 30 as given by a $2 \leftarrow 4$ machine and as given by a $1 \leftarrow 2$ machine.

c) Does there seem to be an easy way to convert from one representation of a number to the other?

Now consider a $1|1 \leftarrow 3$ machine:



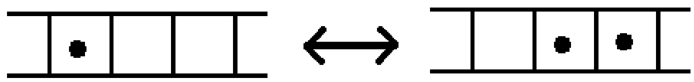
d) Verify that a $1|1 \leftarrow 3$ machine is also a base two machine.

e) Write the numbers 1 through 30 as given by a $1|1 \leftarrow 3$ machine. Is there an easy way to the $1|1 \leftarrow 3$ representation of a number its $1 \leftarrow 2$ representation, and vice versa?

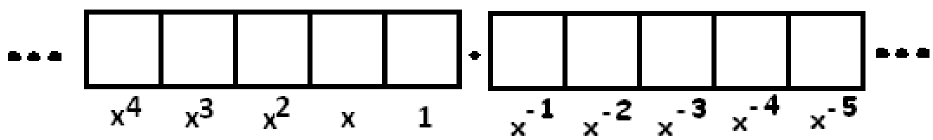
FUN QUESTION: What is the "decimal" representation of the fraction $\frac{1}{3}$ in each of these machines? How does long division work for these machines?

CRAZY MACHINE <u>FOUR</u>: BASE PHI

Consider the very strange machine $1|0|0 \leftrightarrow 0|1|1$. Here two dots in consecutive boxes can be replaced with a single dot one place to the left of the pair and, conversely, any single dot can be replaced with a pair of consecutive dots to its right.



Since this machine can move both to the left and to the right, let's give it its full range of "decimals" as well.



a) Show that, in this machine, the number 1 can be represented as 0.10101010101…. (It can also be represented just as 1 !!)

b) Show that the number 2 can be represented as 10.01.

c) Show that the number 3 can be represented as 100.01

d) Explain why each number can be represented as a in terms of 0s and 1s with no two ones consecutive. (**TOUGH**: Are such representations unique?)

Let's now address the question: What base is this machine?

e) Show that in this machine we need $x^{n+2} = x^{n+1} + x^n$ for all n.

f) Dividing by x^n this tells us that x must be a number satisfying $x^2 = x + 1$. There are two numbers that work. What is the positive number that works?

g) Represent the numbers 4 through 20 in this machine with no consecutive 1s. Any patterns?

RELATED ASIDE??

The Fibonacci numbers are given by:

1, 1, 2, 3, 5, 8, 13, 21, 34, …,

They have the property that each number is the sum of the previous two terms.

In 1972, Edouard Zeckendorf proved that every positive integer can be written as a sum of Fibonacci numbers with no two consecutive Fibonacci numbers appearing in the sum. For example:

$$17 = 13 + 3 + 1$$

and

$$46 = 34 + 8 + 3 + 1$$

(Note: 17 also equals $8 + 5 + 3 + 1$ but this involves consecutive Fibonacci numbers.)

Moreover, Zeckendorf proved that the representations are unique:

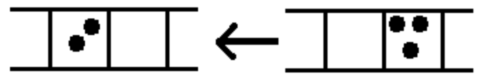
Each positive integer can be written as a sum of non-consecutive Fibonacci numbers in precisely one way.

This result has the "feel" of a base machine at its base.

Is there a way to construct a base machine related to the Fibonacci numbers in some way and use it to establish Zeckendorf's result?

CRAZY MACHINE <u>FIVE</u>: BASE ONE-AND-A-HALF AGAIN

Recall the $2 \leftarrow 3$ machine from page 4. It takes three dots in one box and replaces them with two dots one place to the left.



This machine produces the base $\frac{3}{2}$ representations of numbers. The first twenty-four representations are:

$1 \to 1$	$9 \to 2100$	$17 \to 21012$
$2 \to 2$	$10 \to 2101$	$18 \to 21200$
$3 \to 20$	$11 \to 2102$	$19 \to 21201$
$4 \to 21$	$12 \to 2120$	$20 \to 21202$
$5 \to 22$	$13 \to 2121$	$21 \to 21220$
$6 \to 210$	$14 \to 2122$	$22 \to 21221$
$7 \to 211$	$15 \to 21010$	$23 \to 21222$
$8 \to 212$	$16 \to 21011$	$24 \to 210110 \ \cdots$

a) Any patterns? Why must all the representations (after the first) begin with the digit 2? Do all the representations six and beyond begin with 21? What can you say about final digits? Last two final digits?

b) Notice: 1 dot gives the first one-digit answer
3 dots gives the first two-digit answer
6 dots give the first three-digit answer
9 dots give the first four-digit answer
and so on.

This gives the sequence: **1, 3, 6, 9, 15, 24, 36, 54, 81, 123, …**
ANY PATTERNS?

c) If a_N represents the N th number in the sequence from part b), use the properties of the $2 \leftarrow 3$ machine to establish that:

$$a_{N+1} = \begin{cases} \dfrac{3a_N}{2} & \text{if } a_N \text{ is even} \\ \dfrac{3(a_N+1)}{2} & \text{if } a_N \text{ is odd} \end{cases}$$

<u>HINT</u>: If m dots are needed in the right most box to get an N-digit answer, how many dots are needed to get m dots in the second box?

d) To this day no one knows an explicit formula for a_N. Is it possible to compute a_{1000} without having to compute a_{999} and a_{998} and so on before it?

RELATED ASIDE?? In 1937 L. Collatz posed the following problem …

Choose any positive integer.
If it is even, divide it by two.
If it is odd, triple it and add one.
Either way, this gives a new integer.
Repeat.

For example, starting with 7 we obtain the sequence:

$$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16$$
$$\to 8 \to 4 \to 2 \to 1 \to 4 \to 2 \to 1 \to 4 \to 2 \to 1 \to \cdots$$

Collatz asked:

Does every integer eventually lead to a $4 \to 2 \to 1$ cycle?

No one currently knows … but the conjecture has been checked and found to hold for all numbers from 1 through to $3 \cdot 2^{53}$ (which is about 27000000000000000).

COMMENT: This problem is sometimes called the HAILSTONE PROBLEM because the chain of numbers one obtains seems to bounce up and down for a good while before eventually falling down to 1. For example, the number 27 takes 112 steps before entering a $4 \to 2 \to 1$ cycle, reaching a high of 9232 before getting there!

a) Play with Collatz conjecture. Write a computer program to plot the number of steps it takes for each number to fall into a $4 \to 2 \to 1$ cycle. Any visual structures?

b) Does the mathematics of a $2 \leftarrow 3$ connect with Collatz' problem in some direct way? Ponder and explore.

FINAL THOUGHTS …

Invent other crazy machines …

Invent $a\,|\,b\,|\,c \leftrightarrow d\,|\,e\,|\,f$ machines for some wild numbers a, b, c, d, e, f.

Invent a base half machine

Invent a base negative two-thirds machine

Invent a machine that has one rule for boxes in even positions and a different rule for boxes in odd positions.

Invent a base i machine or some other complex number machine.

How does long division work in your crazy machine?

What is the fraction $\frac{1}{3}$ in your crazy machine?

((OOH! What's $\frac{1}{3}$ in a $2 \leftarrow 3$ machine?))

<u>TOUGH QUESTION</u> … Do numbers have unique representations in your machines or multiple representations?

Go wild and see what crazy mathematics you can discover!

