Finite Differences

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1 Warming up (no formulas on this side!)

Problem 1 For each the following sequences, try to analyze

- What is "the" next number (or two) in "the" sequence?
- What is ~~the~~ **a** pattern that characterizes your sequence? (What types of descriptions count as a pattern?)
- Better still, Find as many patterns as you can describing the sequence.
- For each pattern, can you find other sequences that meet the same pattern? Can you characterize (in some way) the family of sequences?

(A) $3, 7, 11, 15, 19, \ldots$

(B) $13, 6, -1, -8, -15, -23, \ldots$

(C) $-1, 0, 1, 4, 9, 16, \ldots$

(D) $0, 4, 11, 21, 34, \ldots$

(E) $1, 1, 3, 13, 37, 81, \ldots$

(F) $-13, 5, 9, 5, -1, -3, 5, 29, 75, 149, 257, \ldots$

(G) $1, 2, 4, 8, 16, 32, \ldots$

(H) $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\ldots$

2 Finite Differences

Given any sequence of numbers: $a_1, a_2, a_3, a_4, \ldots$

The *sequence of differences* is given by $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots$

It is convenient to write them in the following format

$$\begin{array}{ccccccccccccccc} a_1 & & a_2 & & a_3 & & a_4 & & a_5 & & a_6 & & a_7 \ldots \\ & a_2 - a_1 & & a_3 - a_2 & & a_4 - a_3 & & a_5 - a_4 & & a_6 - a_5 & & a_7 - a_6 & & \ldots \end{array}$$

Example:

$$\begin{array}{ccccccccccccccc} -8 & & -1 & & 0 & & 1 & & 8 & & 27 & & 64 \ldots \\ & 7 & & 1 & & 1 & & 7 & & 19 & & 37 & & \ldots \end{array}$$

Of course, you can take the difference of a sequence of differences, and take the difference of *that* sequence, and so on.

$$\begin{array}{ccccccccccccccc} -8 & & -1 & & 0 & & 1 & & 8 & & 27 & & 64 \ldots \\ & 7 & & 1 & & 1 & & 7 & & 19 & & 37 & & \ldots \\ & & -6 & & 0 & & 6 & & 12 & & 18 & & \ldots \\ & & & 6 & & 6 & & 6 & & 6 & & & \ldots \\ & & & & 0 & & 0 & & 0 & & & & \ldots \end{array}$$

Problem 2 Go back and do this with our example functions. What happens? We should try to come up with some hypotheses, and maybe gather some evidence.

Useful Notation: If we represent our sequence as a_n, we can represent the sequence of differences using the *difference operator*, Δ:

$$\Delta a_n = a_{n+1} - a_n.$$

And, we can call the difference of the difference of a sequence $\Delta(\Delta a_n)$, which can also be written (with some caution) as $\Delta^2 a_n$

3 Working backward

If I know Δa_n, can I reconstruct a_n? (At least, the terms of the sequence).

$$\begin{array}{ccccccccccccccc} ? & & ? & & ? & & ? & & ? & & ? & & ? \ldots \\ & 6 & & 2 & & 0 & & 0 & & 2 & & 6 & & \ldots \end{array}$$

What if I know $\Delta^2 a_n$ is the sequence $3n - 2$?

What if I know $\Delta^3 a_n$ is the constant sequence $-12, -12, -12, -12, -12, \ldots$?

4 Working diagonally

What if I know:

$$
\begin{array}{ccccccccccccc}
1 & & ? & & ? & & ? & & ? & & ? & & ?\ldots \\
& -1 & & ? & & ? & & ? & & ? & & & \ldots \\
& & 2 & & ? & & ? & & ? & & & & \ldots \\
& & & -3 & & ? & & ? & & & & & \ldots \\
& & & & 0 & & 0 & & & & & & \ldots
\end{array}
$$

Do I know the entire (top row) sequence? What additional assumption might allow me to complete the sequence?

Can I find a *formula* for the sequence?

Or how about:

$$
\begin{array}{ccccccccccccc}
a & & ? & & ? & & ? & & ? & & ? & & ?\ldots \\
& b & & ? & & ? & & ? & & ? & & ? & \ldots \\
& & c & & c & & c & & c & & c & & c\ldots
\end{array}
$$

5 The general problem and an approach to a solution

if I know values on one diagonal $d_0, d_1, d_2, \ldots d_n$ (and also that the rows below d_n is entirely 0)

$$
\begin{array}{ccccccccccccc}
d_0 & & ? & & ? & & ? & & ? & & ? & & ?\ldots \\
& d_1 & & ? & & ? & & ? & & ? & & & \ldots \\
& & d_2 & & ? & & ? & & ? & & & & \ldots \\
& & & d_3 & & ? & & ? & & & & & \ldots \\
& & & & 0 & & 0 & & & & & & \ldots
\end{array}
$$

Can I determine the sequence on the top row? Can I express it in a formula in terms of $d_0, d_1, \ldots, d_n$?

5.1 Repertoire method

A very useful idea **that we should verify for ourselves with examples** (and maybe even prove):

If I write the sequence a_n as the sum of two sequences b_n and c_n, then the sequence of differences of a_n is the sum of the two sequence of differences for b_n and c_n. In fact, if $a_n = j \cdot b_n + k \cdot c_n$, then

$$\Delta a_n = j \cdot \Delta b_n + k \cdot \Delta c_n$$

(we could say: the difference operator is *linear*)

Next, can we solve the general problem some special cases? In each of these cases, assume the row below the last row given is entirely 0.

(case 0)

$$\begin{array}{ccccccc} 1 & ? & ? & ? & ? & ? & ?\ldots \end{array}$$

(case 1)

$$\begin{array}{cccccccccccccc} 0 & & ? & & ? & & ? & & ? & & ? & & ?\ldots & \\ & 1 & & ? & & ? & & ? & & ? & & & & \ldots \end{array}$$

(case 2)

$$\begin{array}{cccccccccccccc} 0 & & ? & & ? & & ? & & ? & & ? & & ?\ldots & \\ & 0 & & ? & & ? & & ? & & ? & & ? & & \ldots \\ & & 1 & & ? & & ? & & ? & & ? & & & \ldots \end{array}$$

(case 3)

$$\begin{array}{cccccccccccccc} 0 & & ? & & ? & & ? & & ? & & ? & & ?\ldots & \\ & 0 & & ? & & ? & & ? & & ? & & ? & & \ldots \\ & & 0 & & ? & & ? & & ? & & ? & & & \ldots \\ & & & 1 & & ? & & ? & & ? & & & & \ldots \end{array}$$

(case 4)

$$\begin{array}{cccccccccccccc} 0 & & ? & & ? & & ? & & ? & & ? & & ?\ldots & \\ & 0 & & ? & & ? & & ? & & ? & & ? & & \ldots \\ & & 0 & & ? & & ? & & ? & & ? & & & \ldots \\ & & & 0 & & ? & & ? & & ? & & & & \ldots \\ & & & & 1 & & ? & & ? & & & & & \ldots \end{array}$$

5.2 Pascal's Triangle

You probably already know

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Can we see any connections to finite differences?

6 Convenient notation

It is helpful (but not universal) to use the notation for *falling powers*, that is:

$$x^{\underline{m}} = x(x-1)\cdots(x-m+1)$$

(Rising powers are similarly defined, $x^{\overline{m}} = x(x+1)\cdots(x+m-1)$, but we won't use them here.)

You may also know the expression

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{n(n-1)\cdots(n-m+1)}{m!}$$

in connection with binomial coefficients and Pascal's triangle, but we can also consider them as polynomials in their own right:

$$\binom{x}{m} = \frac{x^{\underline{m}}}{m!} = \frac{x(x-1)\cdots(x-m+1)}{m!}$$

What is $\Delta(x^{\underline{m}})$? What is $\Delta(\binom{x}{m})$? What is $\Delta^k(x^{\underline{m}})$? $\Delta^k(\binom{x}{m})$?

The polynomial $\binom{x}{m}$ is 0 for $x = 0, 1, \ldots, m-1$ and 1 for $x = m$, (Let's verify this!) So we can see how its succession of finite differences will look. This gives a way to resurrect any polynomial from the first (well, 0th) diagonal difference sequence, solving the general problem above.

This approach also gives a nice proof of the recurrence relation:

$$p(x+n) = \binom{n}{1}p(x+n-1) - \binom{n}{2}p(x+n-2) + \ldots + (-1)^{n-1}p(x)$$

for any polynomial of degree *less* than n.

7 Problems that naturally lead to finite differences

Problem 3 Any problem where the sequence of solutions satisfies $a_{n+1} = a_n + P(n)$ where $P(n)$ is a polynomial.

- $a_{n+1} = a_n + k$
- $a_{n+1} = a_n + n$

We might need a starting point a_0 or a_1.

Problem 4 In particular, many summations

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

can be evaluated with this approach, since $S_{n+1} - S_n = \ldots$.

Problem 5 Can we evaluate:

1. $\sum_{k=1}^{n} k, \sum_{k=1}^{n} k^2, \sum_{k=1}^{n} k^3$
2. $\sum_{k=1}^{n} k \cdot (k+3)$
3. $\sum_{k=1}^{n} k^{\underline{3}}, \sum_{k=1}^{n} k^{\underline{4}}$
4. $\sum_{k=1}^{n} \sum_{j=1}^{k} j^2$ (this last one came up in a problem Josh told me yesterday)

Problem 6 (Common) Into how many pieces can a pizza be divided by n straight vertical cuts? (Assume the pizza is essentially 2-dimensional – also convex. And no moving the pieces around between the cuts.)

Problem 7 Into how many pieces can a cake be cut with n straight cuts (not necessarily vertical! The point is that the cake has thickness, so now the shape is 3-dimensional and the cuts are not lines, but planes!)

Problem 8 (More repertoire method than finite differences) The polynomial equation $x^2 - x - 1 = 0$ has the two solutions $\phi = \frac{1+\sqrt{5}}{2} = 1.61803399\ldots$ and $\Phi = -0.61803399\ldots$. The recurrence relation $a_{n+1} = a_n + a_{n-1}$ has many solutions, the most famous being the fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. Show that the geometric sequences $\phi^1, \phi^2, \phi^3, \ldots$ and $\Phi^1, \Phi^2, \Phi^3, \ldots$ satisfy the same recurrence relation. Verify that, if you can find a and b fo rwhich $1 = a\phi^1 + b\Phi^1$ and $1 = a\phi^2 + b\Phi^2$, then the nth Fibonacci number must be $a\phi^n + b\Phi^n$.

8 (more advanced) Contest Problems

Problem 9 (AIME 1992) For any sequence of real numbers $A = (a_1, a_2, a_3, \ldots)$, define ΔA to be the sequence $(a_2 - a1, a_3 - a_2, a_4 - a_3, \ldots)$, whose nth term is $a_{n+1} - a_n$. Suppose that all of the terms of the sequence $\Delta(\Delta A)$ are 1 and that $a_{19} = a_{92} = 0$. Find a_1.

Problem 10 (From the 1995 Polya Team Mathematics Competition) *it will be convenient for us to list the sequences in this round with initial index* 0*: that is, each sequence listed here should be considered to be of the form:* $a_0, a_1, a_2, a_3, \ldots$

(1) The sequence $1, 1, 7, 13, 55, 133, \ldots$ is an example of a sequence that satisfies the recurrence relation

$$a_n = a_{n-1} + 6a_{n-2} \quad \text{for all } n \geq 2.$$

- **(a)** Find all geometric sequences $a_0, a_1, a_2, \ldots$ that
  - **(i)** satisfy the same recurrence relation $a_n = a_{n-1} + 6a_{n-2}$ for all $n \geq 2$.
  - **(ii)** have the first term a_0 equal to 1.
- **(b)** For the sequence $1, 1, 7, 13, 55, 133, \ldots$ listed above, find a closed form expression for the 101^{st} term a_{100} (that is, an expression involving only simple sums, products, and exponentials, without the use of $\sum$ notation or indices).
- **(c)** Prove that there is only one sequence of real numbers satisfying this recurrence relation with both an infinite number of positive terms and an infinite number of negative terms

(2) The sequence $0, 1, 4, 9, 16, 25, \ldots, n^2, \ldots$ is an example of a sequence that satisfies the recurrence relation

$$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} \quad \text{for all } n \geq 3.$$

- **(a)** Find all geometric sequences $a_0, a_1, a_2, a_3, a_4, \ldots$ that
  - **(i)** satisfy the same recurrence relation $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ for all $n \geq 3$.
  - **(ii)** have the first term a_0 equal to 1.
- **(b)** For the general sequence $a_0, a_1, a_2, a_3, \ldots$ satisfying the recurrence relation, find a closed form expression for a_{100} in terms of a_0, a_1, and a_2.
- **(c)** Prove that there are no sequences of real numbers satisfying the recurrence relation with both an infinite number of positive terms and an infinite number of negative terms

(3) Prove that the sequence given by $a_0 = 2$ and, for $n \geq 1$,

$$a_n = \text{ The integer closest to } (5 + 2\sqrt{7})^n$$

satisfies a recurrence relation of the form $a_n = x \cdot a_{n-1} + y \cdot a_{n-2}$ for $n \geq 2$. (For partial credit, find the values for x and y.)