ESTIMATING LARGE NUMBERS (WITHOUT USING A CALCULATOR)!

BRIAN CONREY AND ESTELLE BASOR

1. Introduction

Suppose you are stranded on a desert island, no electricity, and all of your batteries are dead and you need to know how big 100! is. In fact, you are given 5 minutes to give an answer in scientific notation, e.g. 1.6×10^{122}. If you can get the answer to within a factor of 2, a boat will come rescue you.

OK, the 5 minutes starts now!

Time's up, let's collect the answers and see who gets to leave the island.

What did everyone try?

2. Exponents

Here's an approach. Notice that

$$2^{10} = 1024 \approx 1000 = 10^3.$$

This means that

$$2 \approx 10^{0.3}.$$

We could use this to solve the question of the chessboard and grains. One version is this (from Wikipedia).

> When the creator of the game of chess (in some tellings an ancient Indian mathematician) showed his invention to the ruler of the country, the ruler was so pleased that he gave the inventor the right to name his prize for the invention. The man, who was very wise, asked the king this: that for the first square of the chess board, he would receive one grain of wheat (in some tellings, rice), two for the second one, four for the third one, and so forth, doubling the amount each time. The ruler, arithmetically unaware, quickly accepted the inventor's offer, even getting offended by his perceived notion that the inventor was asking for such a low price, and ordered the treasurer to count and hand over the wheat to the inventor. However, when the treasurer took more than a week to calculate the amount of wheat, the ruler asked him for a reason for his tardiness. The treasurer then gave him the result of the

calculation, and explained that it would be impossible to give the inventor the reward.

The total number of grains of wheat on the chessboard will be $2^{64} - 1$. An approximation of 2^{64} is

$$2^{64} \approx (10^{0.30})^{64} = 10^{64 \times 0.30} = 10^{19.2}$$

The name we usually give 10^{19} is "ten quintillion." A calculation found on the internet asserts that there are 5 quintillion grains of sand on all of the beaches in all of the world.

Back to 100!.

If $2 \approx 10^{0.3}$, then

$$5 = \frac{10}{2} \approx \frac{10^1}{10^{0.3}} = 10^{1-0.3} = 10^{0.7}.$$

Notice also that

$$2^2 \times 3^5 = 972 \approx 1000 = 10^3.$$

Thus,

$$3^5 \approx \frac{10^3}{10^{0.6}} = 10^{2.4}.$$

Therefore

$$3 \approx 10^{\frac{2.4}{5}} = 10^{0.48}.$$

Also,

$$\begin{gathered} 4 = 2^2 \approx 10^{0.6} \\ 6 = 2 \times 3 \approx 10^{0.3} \times 10^{0.48} = 10^{0.78} \\ 8 = 2^3 \approx 10^{0.9} \\ 9 = 3^2 \approx 10^{0.96} \end{gathered}$$

We can make a table of n with their exponents:

n	exponent
1	0.00
2	0.30
3	0.48
4	0.60
5	0.70
6	0.78
7	
8	0.90
9	0.96
10	1.00

How about if we fill in 0.84 for the 7 row? Then we could calculate

$$\begin{aligned} 10! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \\ &\approx 10^{0.00} \times 10^{0.30} \times 10^{0.48} \times \cdots \times 10^{0.96} \times 10^{1.00} \\ &= 10^{0.00+0.30+0.48+\cdots+0.96+1.00} = 10^{6.56}. \end{aligned}$$

How big is $10^{6.56}$? Well it's $10^{0.56}$ times a million. And

$$10^{0.56} = 10^{0.48} \times 10^{0.08} = \frac{10^{0.48} \times 10^{0.78}}{10^{0.70}} \approx \frac{3 \times 6}{5} = \frac{18}{5} = 3.6.$$

Thus, our estimate is

$$10! \approx 3.6 \times 10^6.$$

The actual value is

$$10! = 3628800.$$

Use these ideas to estimate the number of positions of Rubik's cube which is

$$\frac{2^{12} \times 3^8 \times 12! \times 8!}{12}.$$

(You should get an answer of around 43 quintillion.)

3. Making the table up to 100

There are certain entries we can fill in and others we cannot. What entries can we fill in? Any number up to 100 whose prime factors are from the set $\{2, 3, 5\}$. This includes 12, 15, 16, 18, 20, 24, 25, 27, 30, 32, 36, 40, 48, 50, 54, 60, 64, 72, 80, 81, 90, 96, and 100.

n	exponent	n	exponent	n	exponent	n	exponent	n	exponent
1	0.00	21		41		61		81	1.92
2	0.30	22		42		62		82	
3	0.48	23		43		63		83	
4	0.60	24	1.38	44		64	1.80	84	
5	0.70	25	1.40	45	1.66	65		85	
6	0.78	26		46		66		86	
7		27	1.44	47		67		87	
8	0.90	28		48	1.68	68		88	
9	0.96	29		49		69		89	
10	1.00	30	1.48	50	1.70	70		90	1.96
11		31		51		71		91	
12	1.08	32	1.50	52		72	1.86	92	
13		33		53		73		93	
14		34		54	1.74	74		94	
15	1.18	35		55		75	1.88	95	
16	1.20	36	1.56	56		76		96	1.98
17		37		57		77		97	
18	1.26	38		58		78		98	
19		39		59		79		99	
20	1.30	40	1.60	60	1.78	80	1.90	100	2.00

What about the rest of the numbers? We should guess the answers and fill them in by interpolation. For example, we have $30 \approx 10^{0.30+0.48+0.70} = 10^{1.48}$ and $32 \approx 10^{1.50}$. So, we should fill in $31 \approx 10^{1.49}$. Also $36 \approx 10^{1.56}$ so we should fill in $34 \approx 10^{1.53}$. Then, $33 \approx 10^{1.515}$ and $35 \approx 10^{1.545}$. Now we are getting more decimal places but notice that

$$33 \times 34 \times 35 \times 36 \approx 10^{1.515+1.53+1.545+1.56} = 10^{6.15}$$

is back to our 2 decimal places.

What if we take a longer stretch? We have $54 \approx 10^{0.30+3\times 0.48} = 10^{1.74}$ and $60 \approx 10^{2\times 0.30+0.48+0.70} = 10^{1.78}$. Now we have to split $1.78 - 1.74 = 0.04$ up into 6 intervals. So we'll get, for example,

$$55 \approx 10^{1.74+(0.04)/6} = 10^{1.74+0.006666\ldots}.$$

Now there are lots of decimal places. However, in our estimation of 100! it is really the product

$$55 \times 56 \times 57 \times 58 \times 59 \times 60$$

that we are interested in. This is approximately 10 to the power

$$(1.74 + \frac{0.04}{6}) + (1.74 + \frac{2\times 0.04}{6}) + (1.74 + \frac{3\times 0.04}{6}) \\ +(1.74 + \frac{4\times 0.04}{6}) + (1.74 + \frac{5\times 0.04}{6}) + (1.74 + \frac{6\times 0.04}{6}).$$

This sum can be rearranged as

$$6 \times 1.74 + \frac{0.04}{6} \times (1 + 2 + 3 + 4 + 5 + 6).$$

Now

$$1 + 2 + 3 + 4 + 5 + 6 = 6 \times \frac{1+6}{2}$$

so that the above is

$$6 \times 1.74 + \frac{7 \times 0.04}{2} = 10.44 + 0.14 = 10.58.$$

In general, if we know that

$$N \approx 10^{A}$$

and

$$N + H \approx 10^{B}$$

then

$$(N+1) \times (N+2) \times \cdots \times (N+H)$$

will be approximately 10 to the power AH plus

$$\frac{(B-A)}{H} + \frac{2(B-A)}{H} + \frac{3(B-A)}{H} + \cdots + \frac{H(B-A)}{H}$$
$$= \frac{(B-A)}{H} \times (1 + 2 + \cdots + H)$$
$$= \frac{(B-A)}{H} \times \frac{H(H+1)}{2}$$
$$= \frac{(B-A)(H+1)}{2}.$$

(Another way to do this calculation is to add up the exponents between the exponent for N and the exponent for $N+H$ inclusive. At N the value is A and at $N+H$ the value is B. Then use the trick of summing an arithmetic progression by taking the number of terms times the average; this gives $(H+1)(A+B)/2$. If we then subtract off the first term $A+(B-A)/H$ we get the same answer as before.)

We apply this to see that the first column will be

$$0.30 + 0.48 + 0.60 + 0.70 + 0.78 + 2 \times 0.78 + \frac{0.12 \times 3}{2} + 0.96 + 1.00$$
$$+2 \times 1.00 + \frac{0.08 \times 3}{2} + 3 \times 1.08 + \frac{0.10 \times 4}{2} + 1.20 + 2 \times 1.20 + \frac{0.06 \times 3}{2}$$
$$+2 \times 1.26 + \frac{0.04 \times 3}{2} = 18.39$$

For the second column we have

$$4 \times 1.3 + \frac{0.08 \times 5}{2} + 1.40 + 2 \times 1.40 + \frac{0.04 \times 3}{2} + 3 \times 1.44 + \frac{0.04 \times 4}{2}$$
$$+2 \times 1.48 + \frac{0.02 \times 3}{2} + 4 \times 1.50 + \frac{0.06 \times 5}{2} + 4 \times 1.56 + \frac{0.06 \times 5}{2}$$
$$= 29.59$$

For the third column we have

$$5 \times 1.60 + \frac{0.06 \times 6}{2} + 3 \times 1.66 + \frac{0.02 \times 4}{2} + 2 \times 1.68 + \frac{0.02 \times 3}{2}$$
$$+4 \times 1.70 + \frac{0.04 \times 5}{2} + 6 \times 1.74 + \frac{0.04 \times 7}{2} = 34.07$$

For the fourth column we have

$$4 \times 1.78 + \frac{0.02 \times 5}{2} + 8 \times 1.80 + \frac{0.06 \times 9}{2} + 3 \times 1.86 + \frac{0.02 \times 4}{2} + 5 \times 1.88 + \frac{0.02 \times 6}{2} = 36.92$$

For the fifth column we have

$$1.92 + 9 \times 1.92 + \frac{0.04 \times 10}{2} + 6 \times 1.96 + \frac{0.02 \times 7}{2} + 4 \times 1.98 + \frac{0.02 \times 5}{2} = 39.2$$

This gives the sum of all five columns

$$18.39 + 29.59 + 34.07 + 36.92 + 39.20 = 158.17$$

Thus, we have deduced that

$$100! \approx 10^{158.17}.$$

Can we put that into scientific notation? To do so we need to know about $10^{0.17}$. Well, we have

$$\frac{3}{2} \approx \frac{10^{0.48}}{10^{0.30}} = 10^{0.18}$$

and

$$\left(\frac{6}{5}\right)^2 \approx \left(\frac{10^{0.78}}{10^{0.70}}\right)^2 = 10^{0.16}.$$

Notice that $3/2 = 1.5$ and $(6/5)^2 = 36/25 = 1.375$, so our estimate is

$$100! \approx 1.4 \times 10^{158}.$$

Another way to do these sums is, for example, when summing the fourth column goes as follows. The sum between 81 and 90 inclusive is $10 \times (1.96 + 1.92)/2$ then between 90 and 96 inclusive is $7 \times (1.96 + 1.98)/2$, then between 96 and 100 inclusive is $5 \times (1.98 + 2.00)/2$. This gives $19.4 + 13.79 + 9.95 = 43.28$. But we have counted the values at 96 and 98 twice. We subtract them off getting a total of $43.28 - 1.96 - 1.98 = 39.20$ as before.

4. Better approximations

We can improve some of our approximations. For example, we know that $10^3 = 1000$ exactly, and

$$1200 = 3 \times 4 \times 100 \approx 10^{0.60} \times 10^{0.48} \times 10^2 = 10^{3.08}.$$

So the exponent changes from 3 to 3.08 as n changes from 1000 to 1200. If we interpolate between 1000 and 1200 in 200 equal steps, each step will have size $\frac{3.08-3}{200} = 0.0004$. This means we should expect that

$$1024 \approx 10^{3+24\times 0.0004} = 10^{3.0096}$$

and since $1024 = 2^{10}$ we expect that

$$2 \approx 2^{3.0096/10} = 10^{0.30096} \approx 10^{0.301}$$

is a good 3 decimal place approximation. Therefore, $5 \approx 10^{0.699}$. Similarly we expect that

$$972 \approx 10^{3-28\times 0.0004} = 10^{2.9888}$$

so that

$$3^5 = 972/4 \approx 10^{2.9888-0.602} = 10^{2.3868}$$

or

$$3 \approx 10^{0.477}.$$

It would be interesting to use these 3-decimal place values to estimate 100!.

5. Thoughts on interpolation

The simplest act of interpolating between exponents might be described as follows. You have a number M that you know is (approximately) 10^A. You have a larger number N which is (approximately) 10^B. Then you interpolate that the average $\frac{M+N}{2}$ is approximately 10 to the power $\frac{A+B}{2}$. Supposing that the original numbers A and B are very accurate, is $\frac{A+B}{2}$ more likely to be an underestimate or an overestimate?

We can reason as follows (let's assume that $M = 10^A$ and $N = 10^B$). We want to compare

$$\frac{M+N}{2} = \frac{10^A + 10^B}{2}$$

with

$$10^{\frac{A+B}{2}}$$

to see if one is always larger that the other. Notice that

$$10^{\frac{A+B}{2}} = 10^{\frac{A}{2}} \times 10^{\frac{B}{2}} = \sqrt{10^A} \times \sqrt{10^B} = \sqrt{M} \times \sqrt{N} = \sqrt{MN}.$$

So, we are trying to compare

$$\sqrt{MN}$$

with

$$\frac{M+N}{2}.$$

Which of these is larger?

It is well known that, for two positive numbers, the arithmetic mean is larger than (or equal to if the numbers are the same) the geometric mean. Can you give a proof?

We can compare the squares of each of these quantities. The square of the geometric mean is just MN. The square of the arithmetic mean is

$$\frac{(M+N)^2}{4}.$$

So, multiplying by 4, we want to compare $4MN$ with $M^2 + 2MN + N^2$. Look that the difference:

$$M^2 + 2MN + N^2 - 4MN = M^2 - 2MN + N^2.$$

This difference is a perfect square, namely $(M - N)^2$, so it's positive, unless $M = N$. In any case, the geometric mean is never larger.

What does this mean about our interpolation? It means that our interpolation is ALWAYS SMALLER than the actual value.

6. Explicit Upper and Lower estimates

Is our estimate of 1.4×10^{158} for 100! too big or too small?

The idea in the last section raises this question. Can we give explicit upper and lower estimates for 100!. In other words can we find two numbers X and Y that aren't too far apart (whatever that means) for which we can say that

$$X < 100! < Y.$$

We know that the 0.30 in $2 \approx 10^{0.30}$ is an underestimate. How do we know this? Because if we use it we are led to

$$1024 = 2^{10} \approx 10^{10 \times 0.30} = 10^3 = 1000.$$

We also know that the process of interpolation always gives an underestimate. If we also had underestimates for the powers of 10 that 3 and 5 are, then our proceedure from the first section would lead to an overall underestimate. Are the values $5 \approx 10^{0.70}$ and $3 \approx 10^{0.48}$ underestimates or overestimates?

They are overestimates, so further ideas would be needed to carry out the suggestion of giving explicit estimates. Please explore!

7. The answer

Mathematica tells us that

$$\begin{aligned} 100! = {} & 93326215443944152681699238856266700490715968264381621 \\ & 46859296389521759999322991560894146397615651828625369 \\ & 7920827223758251185210916864000000000000000000000000 \end{aligned}$$

This is approximately

$$9.3 \times 10^{157}$$