HAVE COLORS –WILL PAINT

(Chromatic Numbers and Other Geometric Combinatorics Delights)

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Imagine that you're completely and utterly free – it's a long day on a vacation island and there's not a single worry or duty in the world. You have lots of blank paper, an unlimited supply of various colored paints and you decide to indulge in painting whatever wild design comes into your mind – stripes, checks, paisley, bird's eye, dog's tooth, herringbone, plaid, polka dots, oh my! Would there be any possible restriction to your exploits? Why should there be any? For example, here's an interesting question:

1. Is it possible to paint every point of the plane using exactly three colors so that every line contains points of exactly two colors?

Or how about some others:

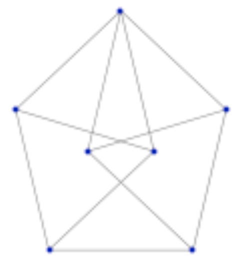
2. Is it possible to color each point on a circle either red or blue in such a way that no three points of the same color form an isosceles triangle? What if instead of just two colors you can use a thousand different colors?

3. How many colors are needed to paint the real number line so that no two points a unit distance apart are painted the same color?

4. How many colors are needed to paint the plane so that no two points a unit distance apart are painted the same color?

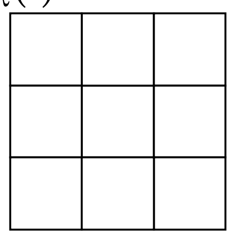
The minimal number of colors which are needed in order to paint all points of the Euclidean *n*-space (that is, space with *n* dimensions) in such a way that no two points a unit distance apart are of the same color is called the *chromatic number* of the space; it is usually denoted by $\chi(n)$.

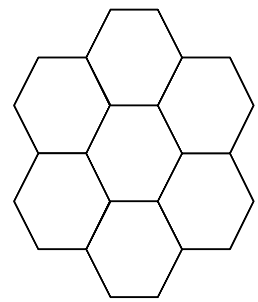
5. In the graph below (called the *Moser Spindle*), the length of each edge is 1. Is it possible to paint the vertices of the graph using three different colors so that no two adjacent vertices are painted the same color? (Note: vertices are called *adjacent* if they are connected by an edge.)

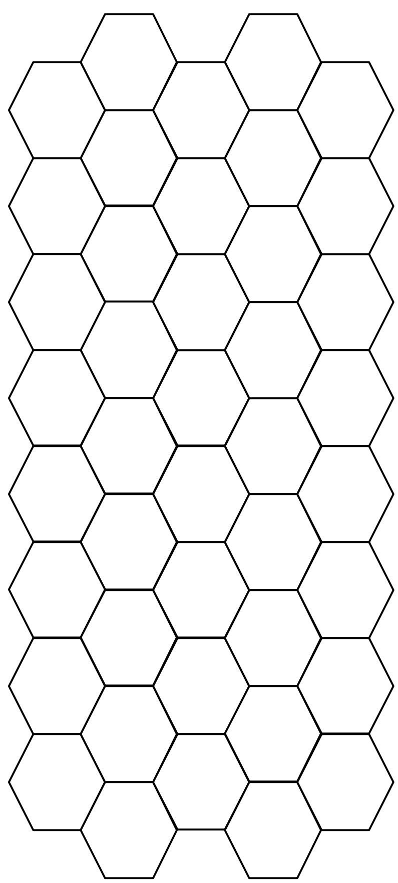


How does painting Moser Spindle help us to decide what are possible values for $\chi(2)$**?**

6. Now let's look at the following diagrams and see how they, too, help in search for that elusive $\chi(2)$**:**







Some closing remarks

The Chromatic Number problem belongs to *geometric combinatorics* (a.k.a. *combinatorial geometry*) which is a relatively new and rapidly growing branch of mathematics. It deals with geometric objects described by a finite set of building blocks, for example, bounded polyhedra and the convex hulls of finite sets of points. Other examples include arrangements and intersections of various geometric objects. Typically, problems in this area are concerned with finding bounds on a number of points or geometric figures that satisfy some conditions, or make a given configuration “optimal” in some sense.

Geometric combinatorics has many connections to linear algebra, discrete mathematics, mathematical analysis, and topology, and it has applications to economics, game theory, and biology, to name just a few.

Problems encountered within geometric combinatorics come in various forms; some are easy to state. Nevertheless, there are lots of problems that are extremely hard to solve, including a great many that remain open despite the efforts of some leading mathematicians. In particular, Borsuk’s problem[1] has not yet been completely solved; to this day, the kissing numbers[2] are known only for dimensions 1, 2, 3, 4, 8, and 24 (these numbers are 2, 6, 12, 24, 240, and 196,560, respectively; surprisingly, the latest one found was the kissing number for dimension 4); as for the chromatic numbers, even $\chi(2)$ has not been found yet!

[1] To find the least number of subsets of smaller diameter in which a set of an *n*-dimensional space could be partitioned.

[2] Two spheres that have exactly one common boundary point are called *kissing* spheres. The largest number of equal spheres kissing a sphere of the same size is called the *kissing number*.