The Teachers' Circle: Farey Fractions & Ford Circles
(A Freshman's Dream)

September 2009

The nth set of *Farey fractions* are all fractions between 0 and 1 with denominator at most n; we collect them in the set F_n. For example,

$$F_1 = \{0, 1\} \qquad \text{and} \qquad F_2 = \left\{0, \tfrac{1}{2}, 1\right\}$$

(here we view 0 as $\frac{0}{1}$ and 1 as $\frac{1}{1}$).

(1) List $F_2, F_3, \ldots, F_7$.

(2) From our construction, the set F_n contains the set F_{n-1}, i.e., all $(n-1)$st Farey fractions are also nth Farey fractions. What do you notice about the fractions that are in F_n but not in F_{n-1}? State as many observations as you can come up with.

(3) Show that if $\frac{a}{b} < \frac{c}{d}$, then

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}\,.$$

(If your proof is algebraic, try to come up with a non-algebraic explanation.) The fraction $\frac{a+c}{b+d}$ is called the *mediant* of $\frac{a}{b}$ and $\frac{c}{d}$.

(4) Let $0 \le \frac{a}{b} < \frac{c}{d} \le 1$ with $bc - ad = 1$. Show that $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in F_n if

- $b, d \le n$ and
- $b + d - 1 \ge n$.

(Hint: if $\frac{a}{b} < \frac{h}{k} < \frac{c}{d}$, write $k = b(ck - dh) + d(bh - ak)$.)

(5) Let $0 \le \frac{a}{b} < \frac{c}{d} \le 1$ with $bc - ad = 1$, and let $\frac{h}{k} = \frac{a+c}{b+d}$. We proved already that $\frac{a}{b} < \frac{h}{k} < \frac{c}{d}$. Show further that

$$bh - ak = 1 \qquad \text{and} \qquad ck - dh = 1\,.$$

(Hint: look at the previous hint.)

(6) Conclude that $F_n \setminus F_{n-1}$ consists of mediants of consecutive fractions in F_{n-1}, and that if $\frac{a}{b} < \frac{c}{d}$ are consecutive then $bc - ad = 1$.

It's time for some polygonal geometry. An integer point (x, y) in the plane is called *visible* if the line segment from the origin to (x, y) contains no other integer point.

(7) For a given positive number n, draw a triangle with vertices $(0, 0)$, $(n, 0)$, and (n, n). Mark all visible points in this triangle. What do you observe? Try to prove your assertions.

Now it's time for some circle geometry. *Ford circles* are constructed as follows: Start with two circles in the plane that touch each other, one tangent to the x-axis at $(0, 0)$ and the other tangent to the x-axis at $(1, 0)$. From here we will create new circles corresponding to Farey fractions; namely, the next circle should be tangent to the x-axis at $(\frac{1}{2}, 0)$ and to the existing two circles, the next two circles tangent to the x-axis at $(\frac{1}{3}, 0)$ and $(\frac{2}{3}, 0)$, respectively, and tangent to their two neighbor circles, etc.

(8) Find a formula for the centers and radii of the Ford circles.

(9)