AIM Teachers' Circle — Partitions

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Here are some of our favorite *partition identities*:

(1) The number of partitions of n into m parts equals the number of partitions of n whose greatest part is m.

(2) The number of partitions of n into at most m parts of size at most k equals the number of partitions of n into at most k parts of size at most m.

(3) The number of partitions of n that are self conjugate equals the number of partitions of n into distinct odd parts.

(4) (Euler) The number of partitions of n into odd parts equals the number of partitions of n into distinct parts.

(5) The number of partitions of n into an even number of odd parts equals the number of partitions of n into distinct parts where the number of odd parts is even.[1]

(6) (Sylvester) The number of partitions of n into k odd parts (repetition allowed) equals the number of partitions of n into k separate sequences of consecutive integers (1-term sequences allowed).

(7) (Bressoud) The number of partitions of n into super-distinct[2] parts equals the number of partitions of n into distinct parts where each even part is greater than twice the number of odd parts.

And here are two problems on *counting* partitions:

(8) How many partitions are there with at most m parts of size at most k? Use this setup to prove the recurrence for binomial coefficients ("Pascal's triangle").

(9) Let c_m denote the number of partitions whose kth part is at most $k-1$ (so their Ferrer's diagram fits into a "staircase"). Show that

$$c_{m+1} = c_0 c_m + c_1 c_{m-1} + \cdots + c_m c_0 \,,$$

where we define $c_0 = 1$. The numbers c_m are called *Catalan numbers* and are given by the formula

$$c_m = \frac{1}{m+1}\binom{2m}{m} .$$

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[1] The same statement holds when we replace the two *evens* with *odd*.
[2] Two parts are *super distinct* if they differ by at least 2.