# EXTEMPORANEOUS TALK ON RANDOM MATRIX THEORY AND SYMMETRIC POWERS

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ABSTRACT. ***Any errors should solely be attributed to the typist, Steven J. Miller***

Want to discuss symmetric poweres of $L$-functions of elliptic curves. Will ignore bad primes, should have a degree two Euler product at the good primes:

$$L(E,s) \;=\; \prod_p^{*} \left(1-\frac{\alpha(p)}{p^s}\right)^{-1}\left(1-\frac{\beta(p)}{p^s}\right)^{-1}. \tag{1}$$

The symmetric cube $L$-function is

$$\begin{aligned} L(\mathrm{sym}^3 E,s) \quad &= \quad \prod_p^{*} \left(1-\frac{\alpha(p)^3}{p^s}\right)^{-1}\left(1-\frac{\alpha(p)^2\beta(p)}{p^s}\right)^{-1} \\ &\quad \cdot \left(1-\frac{\alpha(p)\beta(p)^2}{p^s}\right)^{-1}\left(1-\frac{\beta(p)^3}{p^s}\right)^{-1}. \end{aligned} \tag{2}$$

Note that $\beta(p)=\overline{\alpha(p)}$.

There is a functional equation for $L(\mathrm{sym}^3 E,s)$ relating $s$ to $4-s$. We have a Birch and Swinnerton-Dyer like conjecture:

$$\frac{L(\mathrm{sym}^3 E,\mathrm{center})}{\Omega_{\mathrm{im}}^3\Omega_{\mathrm{re}}}\cdot(2\pi N_E) \;=\; \text{rational with small denominator.} \tag{3}$$

When twist we get

$$\frac{L(\mathrm{sym}^3 E_d,\mathrm{center})}{\Omega/d^2}\cdot(2\pi N_E) \;=\; \text{rational with small denominator.} \tag{4}$$

This should be an orthogonal family. We discretize:

$$\mathrm{Prob}\left(L(\mathrm{sym}^3 E_d,\mathrm{center})<t\right) \;\sim\; \alpha_E t^{1/2}(\log-\mathrm{term}). \tag{5}$$

This leads to

$$\mathrm{Prob}\left(L(\mathrm{sym}^3 E_d,\mathrm{center})=0\right) \;\sim\; (1/d^2)^{1/2}\log-\mathrm{term}. \tag{6}$$

Summing gives that the number of non-vanishing twists of even parity up to $D$ is a power of $\log D$.

| | 11a | 14a (5000) | 15a (4000) |
|---|---|---|---|
| double | 58 | 88 | 83 |
| triple | 1 | 3 | 2 |

Now $L(\mathrm{sym}^3 E, s)$ is not primitive in the CM-case. We should have

$$\frac{L(\mathrm{sym}^3\psi)}{\Omega_{\mathrm{im}}^3}(2\pi)\cdot(\text{rational with small denominator}), \tag{7}$$

with $L(\psi^3)$ of degree 2 (***is this in the numerator?***).

Twists

$$\frac{L(\mathrm{sym}^3\psi_d)}{\Omega/d^{3/2}}\cdot(\text{rational with small denominator}), \tag{8}$$

and then summing gives the number of twists should be $D^{1/4}(\log D)^{\mathrm{power}}$.

In the non-CM case there is roughly 14000 data points. In the CM case there are around $10^5$ or $10^6$ data points, and no triple zeros.