

SHIMURA CORRESPONDENCE AND COMPUTATION OF L-VALUES

ZHENGYU MAO, RUTGERS UNIVERSITY, NEWARK CAMPUS

ABSTRACT. Let $f(z)$ be a new form of weight $2k$ and D a fundamental discriminant. One can associate to $f(z)$ a set of weight $k + \frac{1}{2}$ forms through Shimura correspondence in a generalized sense. We describe some constructions of these half integral weight forms and an application in the computation of L -values $L(f, D, k)$.

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1. INTRODUCTION

Let $f(z)$ be a newform of weight $2k$,

$$f(z) = \sum_{n \geq 1} a_n e(nz) \tag{1.1}$$

(where $e(x) = e^{2\pi i x}$), D a fundamental discriminant and

$$L(f, D, s) = \sum_{n \geq 1} \frac{a_n}{n^s} \left(\frac{D}{n} \right). \tag{1.2}$$

We want to compute $L(f, D, k)$. The idea is to associate to $f(z)$ a weight $k + \frac{1}{2}$ form

$$g(z) = \sum_{n \geq 1} c(n) e(nz). \tag{1.3}$$

Then we relate $L(f, D, k)$ to $c(|D|)$. To carry out this approach we first need a good definition of $g(z)$. First result in that direction is given by Kohnen and Kohnen-Zagier. For $f(z)$ of odd, square-free level N , there is a unique (up to a multiple) weight $k + \frac{1}{2}$ level $4N$ such that

- for $(\ell, N) = 1$, $T_{\ell^2} g = \lambda_{\ell} g$, where $\lambda_{\ell} = a_{\ell} \left(\frac{(-1)^k}{\ell} \right)$;
- in the Kohnen space: $c(n) = 0$ if $(-1)^k n \equiv 2, 3 \pmod{4}$.

Then for fundamental discriminants D satisfying

- $(-1)^k D > 0$;
- $\forall p|N$, $\left(\frac{D}{p} \right) = \omega_p = \pm 1$ (Atkin-Lehner involution)

we have

$$\kappa \frac{c(|D|)^2}{\langle g, g \rangle} = \frac{L(f, D, k)}{\langle f, f \rangle} |D|^{k-\frac{1}{2}} \frac{(k-1)!}{\pi^k}, \quad (1.4)$$

where $\kappa = 2^{-\nu(N)}$ where $\nu(N)$ is the number of prime divisors of N .

2. GENERALIZATIONS

The following is joint with Baruch. For simplicity we assume $N = p$ is prime. Kohnen's formula handles the case for a quarter of these; we want to investigate the fundamental discriminants that do not give rise to vanishing for trivial reasons, namely those with $(-1)^k D < 0$ and $\left(\frac{D}{p}\right) = -\omega_p$. In this case we fix an odd character χ of $(\mathbb{Z}/p\mathbb{Z})^*$. We associate an even character χ on $(\mathbb{Z}/4p\mathbb{Z})^*$, as well as a character χ on $(\mathbb{Z}/4p^2\mathbb{Z})^*$. There is a unique $g'(z)$ in $S_{k+\frac{1}{2}}(4p^2, \chi)$ such that the following is true:

- for $(\ell, N) = 1$, $T_{\ell^2} g' = \lambda'_{\ell} g'$, with $\lambda'_{\ell} = \chi(\ell) a_{\ell} \left(\frac{(-1)^{k+1}}{\ell}\right)$;
- in the Kohnen space: $c'(n) = 0$ if $(-1)^{k+1} n \equiv 2, 3 \pmod{4}$.

For this form $g'(z)$ (unique up to a multiple) we have a similar result:

$$\kappa' \frac{c'(|D|)^2}{\langle g', g' \rangle} = \frac{L(f, D, k)}{\langle f, f \rangle} |D|^{k-\frac{1}{2}} \frac{(k-1)!}{\pi^k}, \quad (2.5)$$

where $\kappa' = (p+1)/2p$.

If $p|D$ and $(-1)^k D > 0$, we still look for a similar result using the form g . Now we would have $\kappa'' = 1$ if $p|D$.

If we relax our conditions a bit, we obtain the following. For m a class modulo p , define canonically a form on $\Gamma_1(4p^2)$ by

$$g'_m(z) = \sum_{n \geq 1} c'_m(n) e(nz) \quad (2.6)$$

where

$$c'_m(n) = \begin{cases} 0 & \text{if } m \not\equiv n \pmod{p} \\ c'(n) & \text{if } m \equiv n \pmod{p}. \end{cases} \quad (2.7)$$

This form is independent of χ , and this is the point of the restriction.

3. GROSS' CONSTRUCTION OF $g(z)$ WHEN $N = p$ AND $k = 2$

What Gross does is he introduces a quaternion algebra:

- B : a quaternion algebra ramified at ∞ and p ;
- R : a fixed maximal order;
- I_i : representatives of right ideal classes.

For example, consider $f_{11A} \rightarrow X_0(11)$. Here $B = B(-1, -11)$ ($a + bi + cj + dk$ with $i^2 = -1, j^2 = -11, ij = k = -ji$), $R = I_1 = \langle 1, i, \frac{1+j}{2}, \frac{i+k}{2} \rangle$ and $I_2 = \langle 2, 2i, \frac{1+2i+j}{2}, \frac{2+3i+k}{2} \rangle$. R_i is the left order of $I_i = \{b : bI_i \subset I_i\}$, and $S_i^0 = \{b \in \mathbb{Z} + 2R_i : \text{Tr}b = 0\}$. In the example above, we have $S_1^0 = \langle 2i, j, i+k \rangle$ and $S_2^0 = \langle 4i, 2i+j, \frac{7i+k}{2} \rangle$. Then

$$\Theta_1[I_i] = \frac{1}{2} \sum_{b \in S_i^0} E(Nb \cdot z) \in S_{3/2}(4p). \quad (3.8)$$

When $L(f, 1, 1) \neq 0$ then

$$g(z) = \sum_i \Theta_i[I_i] \cdot e_i \quad (3.9)$$

where $E_f = (e_i)$ is the eigenvector of the Brandt matrix associated to f . In this case $E_f = \langle 1, -1 \rangle$ and

$$g(z) = \Theta_1[I_1] - \Theta_1[I_2]. \quad (3.10)$$

This is Gross' method, and works well as long as we have non-vanishing.

The reason this construction works involves some correspondence of forms. The basic picture is that we have a quaternion algebra B and f on GL_2 . There is a well-known correspondence (Jaquet-Langlands) between these GL_2 forms f and those E_f on B . This is related to Hecke's basis problem and this case is done by Eichler. We want to associate to f a half-integral weight form g . Instead of going directly to g we first go up to B and then down to g (we go down by a θ correspondence): $g(z) = \langle \Theta_1, E_f \rangle$. When $L(f, 1, 1) = 0$ the image of the θ -correspondence of E_f is 0: $\theta(\pi', \psi) = 0$ (with π for f and π' for E_f).

Work on this has been done by Bochner, Schlze-Pillot, and Pacetti. Here we discuss the work with Rodriguez-Villegas and Toneria. When $L(f, 1, 1) = 0$, from Waldspurger packet theory we can find $g(z)$ in the space of $\theta(\pi' \otimes \chi_\ell, \psi_\ell) \neq 0$ when $L(f, \ell, 1) \neq 0$ (with $\psi_\ell(x) = \exp(2\pi i x / \ell)$). Find an ℓ where $L(f, \ell, 1) \neq 0$, As g is in this θ -correspondence, we have

$$\begin{aligned} g &= \langle E_f \otimes \chi_\ell, \Theta'_\ell \rangle_{\psi_\ell} \\ &= \langle E_f, \Theta_\ell \rangle_\psi, \end{aligned} \quad (3.11)$$

where

$$\Theta_\ell[I_i] = \frac{1}{2} \sum_{b \in S_i^0} \omega_\ell(I_i, b) \exp\left(Nb \cdot \frac{z}{\ell}\right) \quad (3.12)$$

where $\omega_\ell(I_i, b)$ is a weight function. For example, in the case of F37A, we can let $\ell = 5$. In this case $S_3 = \langle 4i, 3i+j, \frac{3i+2j+k}{4} \rangle$, and we can let the weight function $\omega_5(x_1, x_2, x_3) = 0$ if $(5, Nb) = 1, \chi_5(2x_1 + x_2)$ if $2x_1 + x_2 \not\equiv 0 \pmod{5}$ and $\chi_5(x_3)$ otherwise.

Question: This construction gives

$$g(z) = \sum c(n)e(nz); \quad (3.13)$$

however, $g(z)$ is not unique, and the scalar can depend on ℓ . Does there exist an ℓ such that $\gcd_n c(n) = 1$?

We should have

$$L(f, \ell, 1)L(f, D, 1) = \frac{c(|D|)^2}{\sqrt{D\ell}} \frac{\langle f, f \rangle}{\langle E_f, E_f \rangle} \quad (3.14)$$

(as long as $(p, D\ell) = 1$, $\ell > 0$ and $D < 0$; a similar formula was proved by Prasana in the indefinite division algebra case). This has been verified experimentally in many cases. The choices of ℓ and D can be fairly arbitrary. It gives arithmetic properties of the L -functions, as the $c(|D|)^2$ are squares of integers. It tells us that

$$\frac{L(f, \ell, 1)L(f, D, 1)}{\langle f, f \rangle} \sqrt{D\ell} \in \mathbb{Q}. \quad (3.15)$$

Case $g'(z)$: We can find g' in $\theta(\pi', \psi^{-1}) = \theta(\pi' \otimes \chi_\ell, \psi_{-\ell})$ ($\ell < 0$). Find $\ell < 0$ so that $L(f, \ell, 1) \neq 0$.

$$g' = \langle E_f, \Theta_\ell \rangle \quad (3.16)$$

where

$$\Theta_\ell[I_i] = \frac{1}{2} \sum_{b \in S_i^0} \omega_{-\ell}(I_i, b) \omega_p(I_i, b) e\left(Nb \cdot \frac{z}{-\ell}\right) \in S_{3/2}(4p^2, \chi) \quad (3.17)$$

(with $\ell \equiv 3 \pmod{4}$). We needed to introduce the odd function $\omega_p(I_i, b)$ (multiplies with the other odd to give an even). We obtain a similar identity as before. Numerics suggest this is true. Next step, of course, is to provide a proof!

4. QUESTIONS AND FUTURE WORK

- (1) For the higher weight case, all we need to do is introduce another weight function

$$\Theta_\ell = \sum \omega_\infty \omega_\ell \omega_p e\left(Nb \cdot \frac{z}{-\ell}\right). \quad (4.18)$$

- (2) What happens in the composite level case?
- (3) What happens in the even level case?
- (4) Some forms do not have Jacquet-Langlands correspondence. Fortunately there are not too many (56 out of 500,000).