

# L-FUNCTIONS, RANKS OF ELLIPTIC CURVES AND RANDOM MATRIX THEORY

DISCUSSION SESSION: TUESDAY, JULY 10TH, 2007

ALL

ABSTRACT. Problem session on topics to study (moderated by Brian Conrey).  
*All errors should be attributed solely to the typist, Steven J. Miller*

## 1. STATISTICS OF $N$ CONSECUTIVE EIGENVALUES

*Missed beginning as it takes my computer awhile to boot up.*

Mike Rubinstein: Instead of looking at  $U(N)$  for finite  $N$ , let  $M$  tend to infinity and look at  $N$  consecutive normalized eigenvalues of  $U(M)$ .

David Farmer: There's no way that this will be what we want. In the large  $M$  limit we have a characteristic function of whether or not overlap. Nina and Jon's picture: hasn't converged to the Gaussian, won't appear. Will throw away all the lower order terms.

Mike R: Is that a theorem?

David F: Take a huge matrix, segment of its characteristic polynomial will be exactly Gaussian and not skewed Gaussian.

Mike R: So then David thinks we can easily check w/ a single statistic, numerical statistic.

David F: yes.

Mike R: Eigenvalues will still repel. Jon?

Jon Keating: You said  $2 \times 2$  matrices. There are some statistics where answer is extremely close to infinite. Local statistics largely independent of size of matrix block. Long range statistics.

David F: leading order close, not lower order. Lower order will fall apart instantly if  $M$  is not the right size.

Mike R: this is the better model to fit what we're doing with  $L$ -functions. (The mixed Hadamard and primes model).

## 2. WEYL MEASURE: IS IT RMT OR THE UNDERLYING DISTRIBUTIONS THAT ARE SIGNIFICANT FOR $L$ -FUNCTIONS

David Farmer: In Keating's talk he showed calculations involving Weyl integration formula. Everything done in subject only uses that measure. Now, if had been told that measure but not where it came from, would you think RMT is the right thing as to where it came from, or something else? Only the *measure* on the eigenvalues we use; might be missing something (only using measure).

Mike R: What about function field analogue, different families (unitary, symplectic, orthogonal). How are the  $L$ -functions choosing?

David F: sounds natural, but not necessarily right.

Jon Keating: famous: arrival times of bus in Mexican town, spacing b/w parked cars in London. Same measure, no matrix. Comes from an entropy formula. Appears elsewhere.

David F: take  $N$  points at the origin, Brownian motion to the unit circle without allowing crossing. Unitary statistics. What if someone offered this as the hypothesis as the behavior of the  $L$ -functions.

### 3. PACKAGING TWISTS BY CUBIC DIRICHLET CHARACTERS

Hershy Kisilevsky:  $L$ -values from cubic twists. Envious of Rubinstein who can use half-integral weights to get millions of computations. Find some way to package things.

Henri Darmon: Might be a paper in some special situations.

Mike Rubinstein: Maybe fast Fourier technique for the ternary calculations. Maybe  $D^\epsilon$  for a single computation on average.

Brian Conrey: to calculate one value is  $\sqrt{T}$  steps, but to calculate  $T$  of them it takes about  $T^{1+\epsilon}$  (O-S). Maybe if we organize it to recognize things being repeated we can save a  $\sqrt{D}$ .

### 4. MANY ( $10^{12}$ ) QUADRATIC TWISTS

Mike R: This would be another project, in the back of my mind. Today might be possible to do  $10^{10}$  to maybe  $10^{12}$  (pushing it).

Jack Fearnley: we're at about  $10^6$ .

Mike R: It's on a logarithm scale. There are some secondary terms which we will see a bit of an improvement on.

Brian C: certain asymptotics that we just won't be able to see. Many problems trying to modify  $X^A \log^B X$ , tough to get at our level of data.

### 5. GIVING MEANING TO $N = -1/2$ IN THE MOMENT FUNCTION

Mike Rubinstein: Jon's question on giving meaning to  $N = -1/2$  in the moment function. In the moment function when you write in terms of the Barnes  $G$ -function it is meromorphic in  $N$ , first pole at  $N = -1/2$ .

Brian C: In

$$M_U(N, z) = \int_{U(N)} |\Lambda(1)|^2 d\mu, \quad (5.1)$$

we get a formula involving the Barnes  $G$ -function. What do the poles in  $N$  mean?

Mike R: could be similar to the meaning we have from  $\zeta(s)$  where the sums are meaningless.

Brian C: Paul has a formula for

$$\int_{U(N)} |\Lambda(1)|^{2k-r} |\Lambda'(1)|^{2r} d\mu \quad (5.2)$$

for integers  $r$  and  $k$ : analytic continuation? (Note: may have the wrong exponents for the  $\Lambda$ 's.)

Paul: If you fix  $k$ , can only evaluate for a finite number of  $r$ 's. Integer points under a line in the  $(r, k)$ -plane. Know it is not the correct formula if continue. Use finitely many points, analytically continue above the line, but can get to that point by analytically continuing horizontally (which know is correct), but get two different answers.

David F: there is at least one example where replacing factorials with Gamma functions doesn't give the right thing (when going from integers to non-integers). Always say there are formulas that interpolate. There are so many of them that surely some are wrong. Here is an example. I would like to understand when it is okay to interpolate, when it isn't, should we be bothered by this pole at  $-1/2$ ?

Mike R: related to the question: can we develop an analytic theory for the full moments. Leading term is just  $a_k g_k$ . Can we develop heuristics (proofs are beyond us) for the analytic theory of the full moment:

$$\int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^{2k} dt = \int_0^T P_k \left( \log \frac{t}{2\pi} \right) dt + O(T^{1/2+\epsilon}). \quad (5.3)$$

Here we can do it for  $k$  an integer, want to interpolate for all  $k$ . We need a difference approach to the moments that will allow us to work it out. Maybe a refined version of the hybrid formula and the independence hypothesis.

## 6. INDEPENDENCE IN THE HYBRID FORMULAS

Mike Rubinstein: Hybrid formulas for  $L$ -functions, use these formulas to do the moments, assume the truncated Euler product and the truncated Hadamard product are independent in the sense that when you do the moments you can do each piece separately. Only gives the leading term but not the lower terms. How independent are they (statistical tests)? The fact that it gives the moments is a confirmation. Can we attach a statistical value to how independent on the two factors.

Jon Keating: independence is proved for the first few moments. Can prove it splits because we know what the moments are.

Mike R: doesn't mean it is independent, just an identity consistent with being independent. Maybe try and correct for the fact that they are not completely independent, maybe would help with the previous question.

Brian C: use the hybrid model to find lower order terms.

## 7. CONSTRUCTING CERTAIN SIEGEL MODULAR FORMS

Brian Conrey: Construct Siegel modular forms of degree 2 and weight 2 or 3 whose spinor  $L$ -function is primitive (means doesn't split into a product of lower  $L$ -functions).

Audience and Conrey: Upper bound for dimension computed for these spaces up to 23 and we have enough examples to exhaust (??). In principle can produce all the examples and test. As soon as given a bound can try to meet it. David F: for which levels do we know the space of cusp forms? Answer: don't have exact formula, have upper bound, generate lifts (more or less experimentally), as soon as have as many linearly independent lifts as allowed we are done. In some cases there are known formulas ( $\Gamma_0(p)$  and higher weights, starting at weight 5). Hope is for  $p$  large. Conrey: what if someone told you one exists for  $p = 101$ : can you find it? Answer: don't know, think would be hard. David F: If it factors, as it's degree 4 it factors into either two of degree 2 or one of degree 1 and one of degree 3. Know bounds of conductors: test whether or not it factors into degree 1 or degree 2 (know candidates, check by brute force).

## 8. SPECIAL VALUES OF QUADRATIC TWISTS $D$ OF RANKIN-SELBERG ON $GL_2 \times GL_2$

Brian Conrey: This is another degree 4 thing. This could possibly produce a bunch of degree 4 things that vanish.

Mark Watkins: have with symmetric powers. Doesn't seem to match with RMT. It seems that the ones for which RMT produces more twists that vanish actually have less twists that vanish.

## 9. ABELIAN SURFACES

Henri Darmon (and audience and Brian Conrey): Look at twists by Dirichlet characters of order  $\ell$  (a prime) of the  $L$ -function of an abelian surface. DFK (David, Fearnley and Kisilevsky) conjectures for elliptic curves that  $\ell \geq 7$  implies only finitely many twists vanish. What about abelian surfaces?

Brian Conrey (and audience): need to know size of  $L$ -function, size of  $D$  in special value. Degree 4 thing. Are these standard  $L$ -functions? Are they primitive? Can compute the value of the  $L$ -function, if small enough declare it to be zero.

Henri Darmon: when twist pick up a  $D^2$ ?

Brian Conrey: weight  $k$  get  $1/|D|^{(k-1)/2}$ . This goes to  $1/|D|^{(k-1)/4}$  when take the square root. If  $\frac{k-1}{4} \leq 1$  expect infinitely many non-trivial vanishings, and if  $\frac{k-1}{4} > 1$  then only finitely many. If can answer the question as to what the power of  $D$ , would lead to the conjecture.