SHINTANI LIFTS, p-ADIC FAMILIES AND DERIVATIVES OF QUADRATIC TWISTS

HENRI DARMON, MCGILL UNIVERSITY

ABSTRACT. ***Thanks to Steven J. Miller for texing these notes. Any inaccuracies or obscurities of exposition should be bamed on the speaker!***

1. INTRODUCTION

Let N be an odd, square-free integer, $S_k(\Gamma_0(N))$ be the space of cusp forms of weight k on $\Gamma_0(N)$, $k \equiv 2 \bmod 4$, and $S_{(k+1)/2}(\Gamma_0(4N))$ the space of forms of weight $(k+1)/2$ in the Kohnen subspace.

Theorem 1.1 (Shimura-Kohnen)**.** *The spaces $S_k^{\rm new}(\Gamma_0(N))$ and $S_{(k+1)/2}^{\rm new}(\Gamma_0(4N))$ are isomorphic as modules over the Hecke algebra.*

Given an eigenform $f \in S_k^{\rm new}(\Gamma_0(4N))$ there is a corresponding $g \in S_{(k+1)/2}^{\rm new}(\Gamma_0(4N))$, which is unique up to multiplication by a non-zero scalar. Let $g(q) = \sum \sum_{D>0} c(D) q^D$ denote its Fourier expansion.

Theorem 1.2 (Kohnen)**.** *We have*

$$c(D)^2 \;=\; \begin{cases} \lambda_g \sqrt{D}^{k-1} L(f, \chi_{-D}, k/2) & \text{if } -D \equiv 0, 1 \bmod 4,\ \chi_{-D}(\ell) = \omega_\ell \text{ if } \ell | N \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

The question which motivates the present talk is the following. By Kohnen's theorem, the central critical values of quadratic twists of f are packaged into a modular generating series. Can we do something similar for first derivatives? Is there a way to package them into a generating series with a modular interpretation?

This is the motivation. We will discuss a partial result saying something in this direction (joint work in progress with Gonzalo Tornaria). We exploit p-adic families of modular forms.

2. OVERVIEW OF p-ADIC FAMILIES

Let f be a modular form of weight 2 on $\Gamma_0(N)$ which is associated to an elliptic curve E. Fix a prime p dividing N (recall that N is a square-free integer, so $p || N$). This implies $a_p(f) = \pm 1$ (the negative of the eigenvalue of the Atkin-Lehner involution), and thus $a_p(f)$ is a p-adic unit.

Fix a p-adic domain $U \subset \mathbb{Z}_p$.

Definition 2.1. *A p-adic family of modular forms is a formal q-series $\sum a_n(k) q^n$, where the $a_n(k)$ are p-adic analytic functions of $k \in U$, with the property that $f_k := \sum a_n(k) q^n$ is a normalised eigenform of weight k on $\Gamma_0(N)$, for all $k \in U \cap \mathbb{Z}^{\geq 2}$.*

Example 2.2. *There are the following basic examples of p-adic families of modular forms:*

- **Eisenstein series**. *Let*

$$E_k^* \;=\; \zeta^*(1-k) + \sum_{n=1}^{\infty} \sigma_{k-1}^*(n) q^n, \tag{2.2}$$

with $\sigma_{k-1}^*(n) = \sum_{d|n,(p,d)=1} d^{k-1}$ *(the* ζ^* *means we remove the factor corresponding to* p*).*

- **Binary theta series**. *Let* ψ *be a Hecke character of an imaginary quadratic field* K*, of* ∞ *type* $(1,0)$,

$$\theta_k \;=\; \sum_{\substack{a \lhd \mathcal{O}_K \\ (p,a\bar{a})=1}} \psi(a)^{k-1} q^{a\bar{a}}. \tag{2.3}$$

The following theorem of Hida shows that these families are in some sense quite ubiquitous.

Theorem 2.3 (Hida)**.** *There exists a unique* p*-adic family* f_k *satisfying* $f_2 = f$*.*

Remark 2.4 (Important)**.** *For all* $k \in U \cap \mathbb{Z}^{\geq 2}$*, We have* $f_k \in S_k(\Gamma_0(N))$*, though these are not necessarily newforms. For* $k > 2$*, they are* not *new at* p*, but are new at all other primes dividing* $N := pM$*. Thus there exists a newform* $f_k^{\#} \in S_k^{\mathrm{new}}(\Gamma_0(M))$ *with the same Hecke eigenvalues at* $\ell \neq p$*.*

Let $g_k \in S_{(k+1)/2}^{\mathrm{new}}(\Gamma_0(4M))$ be the form which corresponds to $f_k^{\#}$ under the Shimura-Kohnen correspondence:

$$g_k \;=\; \sum_{D>0} c(D,k) q^D. \tag{2.4}$$

These forms have potentially twice as many non-vanishing Fourier coefficients. This is because the level has dropped.

There are two types of D for which, a priori, $c(D,k)$ could be non-zero:

- Type I: D such that $\chi_{-D}(\ell) = \omega_\ell$ for all $\ell | N$: for these $L(f,-D,s)$ has sign 1 it its functional equation and $c(D)$ encodes its central critical value $L(f,-D,1)$;
- Type II: D such that $\chi_{-D}(\ell) = \omega_\ell$ for all $\ell | M$ but $\chi_{-D}(p) = -\omega_p$: for these, $L(f,-D,s)$ has sign -1 it its functional equation and it becomes natural to consider $L'(f,-D,1)$.

Note that the g_k are only defined up to a non-zero scalar. We will normalise them by setting a coefficient equal to 1. There is a fact (due to Glenn Stevens) which asserts that there is a Δ_0 (of type I) such that $c(\Delta_0,k) \neq 0$ for all k in a p-adic neighborhood of 2, and in particular $c(\Delta_0) \neq 0$. By dividing by that coefficient, we can define

$$\widetilde{c}(D,k) \;=\; \frac{1-\chi_{-D}(p)a_p(k)^{-1}p^{\frac{k-2}{2}}}{1-\chi_{-\Delta_0}(p)a_p(k)^{-1}p^{\frac{k-2}{2}}} \cdot \frac{c(D,k)}{c(\Delta_0,k)}. \tag{2.5}$$

These are defined for $k \in U \cap \mathbb{Z}^{>2}$, and we can show

$$\widetilde{c}(D,k) \;=\; \frac{c(p^2D,k)}{c(p^2\Delta_0,k)}. \tag{2.6}$$

Theorem 2.5 (Hida, Stevens)**.** *The functions* $k \mapsto \widetilde{c}(D,k)$ *extend to analytic functions in a neighborhood of* $k=2$*.*

If the discriminant D is of type II, then $\widetilde{c}(D,2) = 0$. Our main theorem is the following:

Theorem 2.6 (Tornaria-Darmon)**.** *Let* $-D$ *be a type II discriminant. Then there exists a point* $P_D \in E^{-D}(\mathbb{Q}) \otimes Q$ *(the Mordell-Weil group attached to the elliptic curve* E^{-D}*) such that*

(1) $\frac{d}{dk}\widetilde{c}(D,k)_{k=2} = \log_p(P_D)$ *(here* $\log_p : E^{-D}(\mathbb{Q}_p) \to \mathbb{Q}_p$ *is the* p*-adic formal group logarithm).*
(2) *The point* P_D *is of infinite order if and only if* $L'(f,-D,1) \neq 0$.

One way of stating this result is the following. After normalising g_k (and applying a Hecke operator U_{p^2} to it) we have the following "first order expansion" of g_k in a neighbourhood of weight $3/2$:

$$g_k = \sum_{D \text{ type I}} c(D)q^D + (k-2) \sum_{D \text{ type II}} \log_p(P_D)q^D + O((k-2)^2). \tag{2.7}$$

Question 2.7. *Natural questions:*

(1) *What can we say about the order of vanishing at* $k = 2$ *of* $c(D,k)$*? Presumably this is related to the rank of* E^{-D}.
(2) *What can we say about leading terms?*
(3) *Is there a non-*p*-adic version?*
(4) *Is this formula useful (for the types of questions considered in this workshop)? The original formula of Kohnen and Waldspurger helps analyze a large number of non-vanishing of quadratic twists. Can we do something similar for first derivatives?*

To amplify on the last question: Consider a k which is p-adically close to 2, say $k = 2 + (p-1)p^M$. We can look at $\widetilde{c}(D,k)$. If D is of type II, we know $\widetilde{c}(D,k) \equiv 0 \bmod p^{M+1}$. If $\widetilde{c}(D,k) \not\equiv 0 \bmod p^{2M}$ then $L'(f,-D,1) \neq 0$. So if we are looking for non-vanishing twists of derivatives, we can search instead for coefficients of forms of higher half integral weight that are not divisible by p^{2M}.

The key ingredient in the proof of the main theorem are

- a formula (due to Kohnen) relating integrals of modular forms to products of the coefficients: $c(D,k)c(\Delta,k)$ is a combination of geodesic cycle integrals attached to f_k and binary quadratic forms of discriminant $D\Delta$;
- The theory of Stark-Heegner points.