L-FUNCTIONS, RANKS OF ELLIPTIC CURVES AND RANDOM MATRIX THEORY

Finite conductor models for zeros of elliptic curves

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Orthogonal Random Matrix Models

RMT: $2N$ eigenvalues, in pairs $e^{\pm i\theta_j}$, probability measure on $[0, \pi]^N$:

$$d\epsilon_0(\theta) \ \propto \ \prod_{j<k} (\cos\theta_k - \cos\theta_j)^2 \prod_j d\theta_j.$$

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Independent Model:

$$\mathcal{A}_{2N,2r} \;=\; \left\{ \begin{pmatrix} I_{2r\times 2r} & \\ & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$

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Interaction Model:

Sub-ensemble of $SO(2N)$ with the last $2r$ of the $2N$ eigenvalues equal $+1$:

$$d\varepsilon_{2r}(\theta) \;\propto\; \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j (1-\cos\theta_j)^{2r} \prod_j d\theta_j,$$

with $1 \;\leq\; j,k \;\leq\; N-r$.

1-Level Density

L-function $L(s, f)$: by RH non-trivial zeros $\frac{1}{2} + i\gamma_{f,j}$.

C_f: analytic conductor; $\varphi(x)$: compactly supported even Schwartz function.

$$D_{1,f}(\varphi) \;=\; \sum_j \varphi\left(\frac{\log C_f}{2\pi}\gamma_{f,j}\right)$$

- individual zeros contribute in limit
- most of contribution is from low zeros

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Katz-Sarnak Conjecture:

$$\begin{aligned} D_{1,\mathcal{F}}(\varphi) \;&=\; \lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} D_{1,f}(\varphi) \;=\; \int \varphi(x)\rho_{G(\mathcal{F})}(x)dx \\ &=\; \int \widehat{\varphi}(u)\widehat{\rho}_{G(\mathcal{F})}(u)du. \end{aligned}$$

Random Matrix Models and One-Level Densities

Fourier transform of 1-level density:

$$\hat{\rho}_0(u) = \delta(u) + \frac{1}{2}\eta(u).$$

Fourier transform of 1-level density (Rank 2, Independent):

$$\hat{\rho}_{2,\text{Independent}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right].$$

Fourier transform of 1-level density (Rank 2, Interaction):

$$\hat{\rho}_{2,\text{Interaction}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right] + 2(|u| - 1)\eta(u).$$

Comparing the RMT Models

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right) \ = \ \int \varphi(x)\rho_{\mathcal{G}}(x)dx + \ r\varphi(0$$

where

$$\mathcal{G} \ = \ \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

Confirm Katz-Sarnak, B-SD predictions for small support.

Supports Independent and not Interaction model in the limit.

Testing Random Matrix Theory Predictions

Know the right model for large conductors, searching for the correct model for finite conductors.

In the limit must recover the independent model, and want to explain data on:

1. **Excess Rank:** Rank r one-parameter family over $\mathbb{Q}(T)$: observed percentages with rank $\geq r + 2$.

2. **First (Normalized) Zero above Central Point:** Influence of zeros at the central point on the distribution of zeros near the central point.

Excess Rank

One-parameter family, rank r over $\mathbb{Q}(T)$.

Density Conjecture (Generic family) $\Longrightarrow$ 50% rank r, r+1.

For many families, observe

Percent with rank r	$\approx$	32%
Percent with rank r+1	$\approx$	48%
Percent with rank r+2	$\approx$	18%
Percent with rank r+3	$\approx$	2%

Problem: small data sets, sub-families, convergence rate $\log(\text{conductor})$.

Data on Excess Rank

$$y^2 + y = x^3 + Tx$$

Each data set 2000 curves from start.

t-Start	Rk 0	Rk 1	Rk 2	Rk 3	Time (hrs)
-1000	39.4	47.8	12.3	0.6	<1
1000	38.4	47.3	13.6	0.6	<1
4000	37.4	47.8	13.7	1.1	1
8000	37.3	48.8	12.9	1.0	2.5
24000	35.1	50.1	13.9	0.8	6.8
50000	36.7	48.3	13.8	1.2	51.8

Last set has conductors of size 10^{17}, but on logarithmic scale still small.

RMT: Theoretical Results ($N \to \infty$, Mean $\to 0.321$)

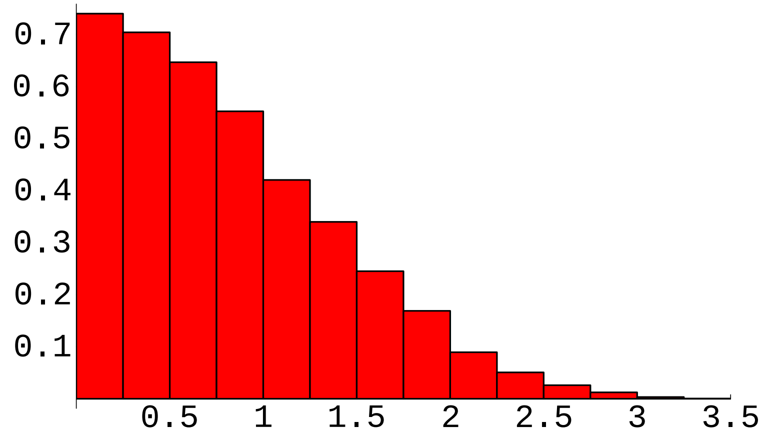

Figure 1a: 1st norm. evalue above 1: 23,040 SO(4) matrices
Mean $= .709$, Std Dev of the Mean $= .601$, Median $= .709$

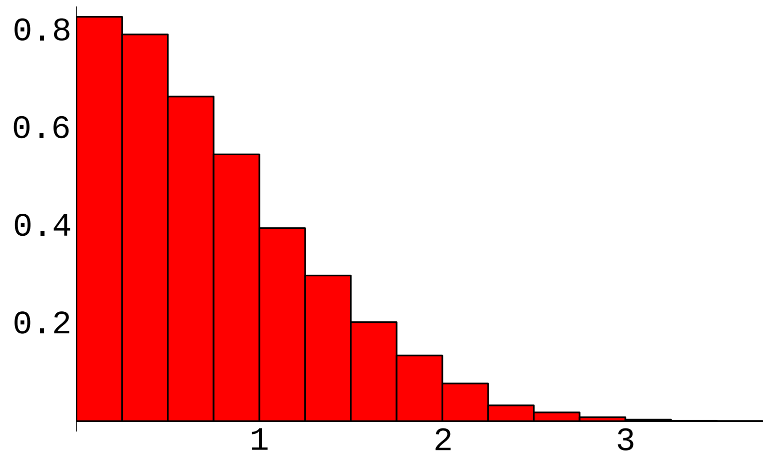

Figure 1b: 1st norm. evalue above 1: 23,040 SO(6) matrices
Mean $= .635$, Std Dev of the Mean $= .574$, Median $= .635$

RMT: Theoretical Results ($N \to \infty$)

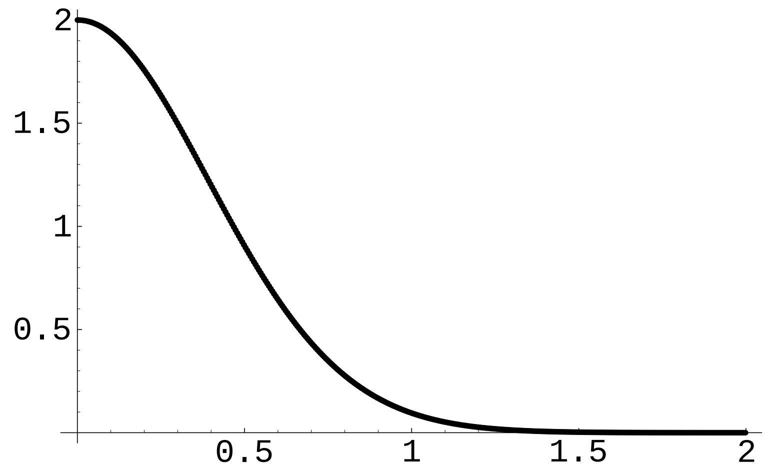

Figure 1c: 1st norm. evalue above 1: SO(even)

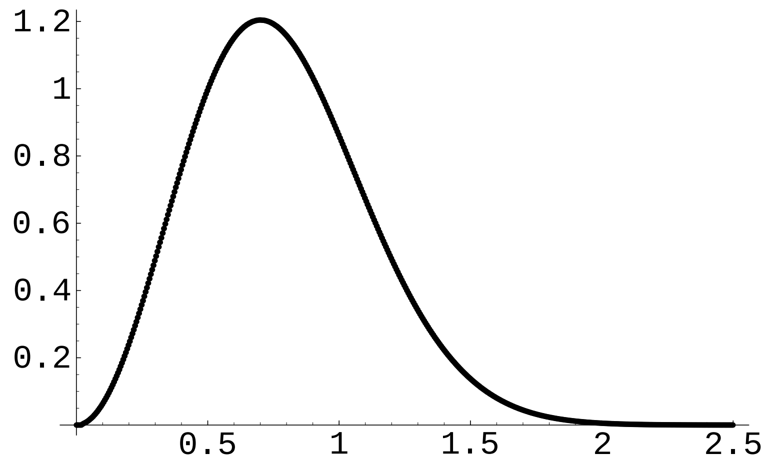

Figure 1d: 1st norm. evalue above 1: SO(odd)

Rank 0 Curves: 1st Normalized Zero above Central Point

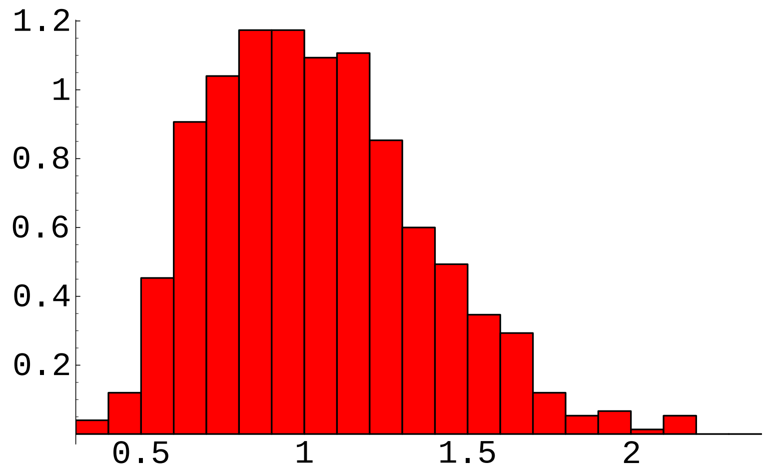

Figure 2a: 750 rank 0 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [3.2, 12.6]$, median $= 1.00$ mean $= 1.04$, $\sigma_\mu = .32$

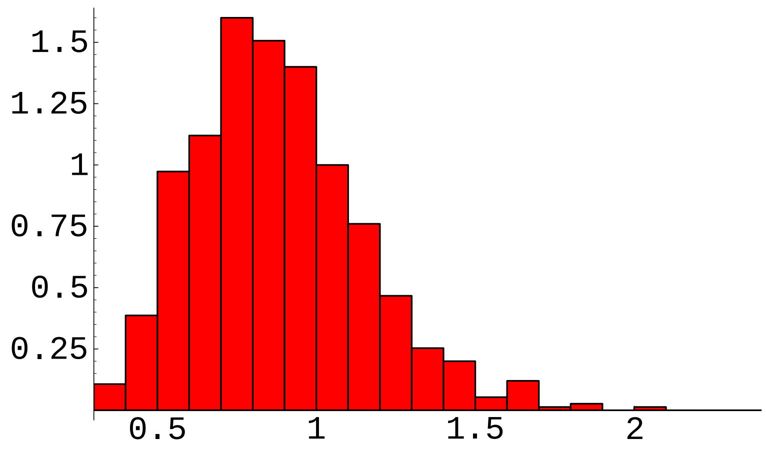

Figure 2b: 750 rank 0 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [12.6, 14.9]$, median $= .85$, mean $= .88$, $\sigma_\mu = .27$

Rank 2 Curves: 1st Norm. Zero above the Central Point

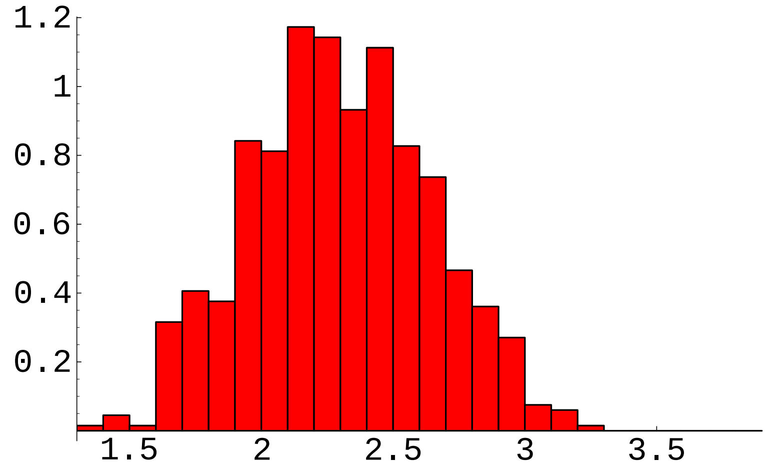

Figure 3a: 665 rank 2 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [10, 10.3125]$, median $= 2.29$, mean $= 2.30$

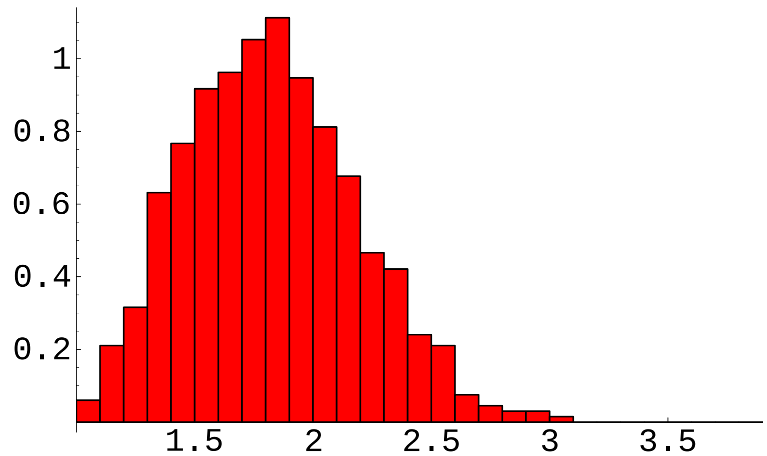

Figure 3b: 665 rank 2 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [16, 16.5]$, median $= 1.81$, mean $= 1.82$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

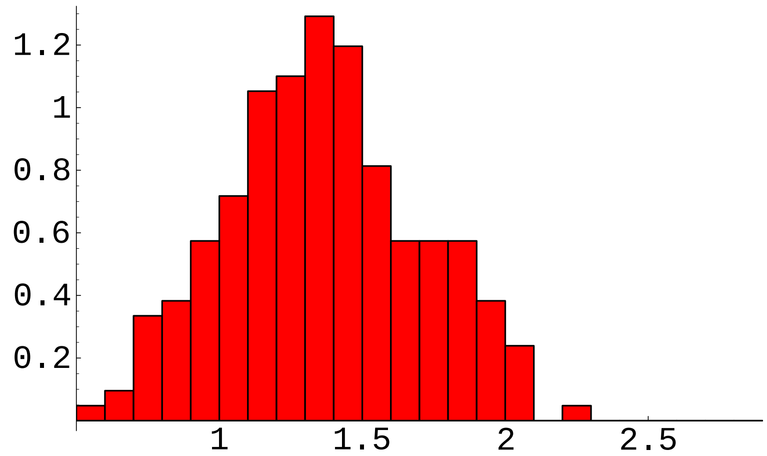

Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

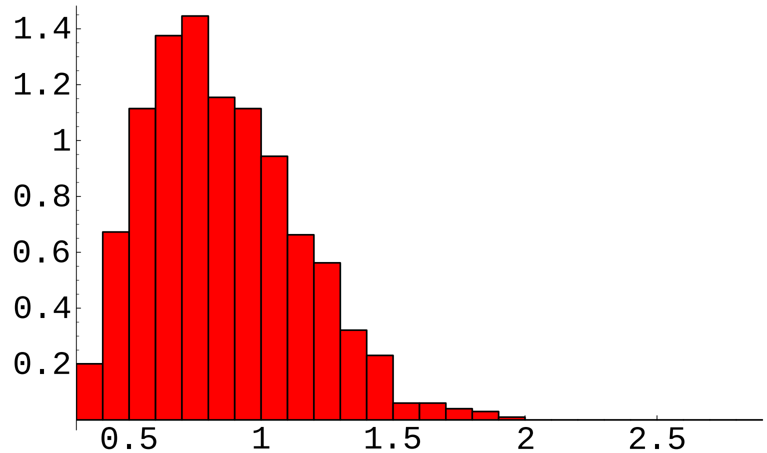

Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Rank 2 Curves: 1st Norm Zero: one-param of rank 0 over $\mathbb{Q}(T)$

first set $\log(\text{cond}) \in [15, 15.5)$; second set $\log(\text{cond}) \in [15.5, 16]$. Median $\widetilde{\mu}$, Mean μ, Std Dev (of Mean) σ_μ.

Family	$\widetilde{\mu}$	μ	σ_μ	**Number**	$\widetilde{\mu}$	μ	σ_μ	**Number**
1: [0,1,3,1,T]	1.59	1.83	0.49	8	1.71	1.81	0.40	19
2: [1,0,0,1,T]	1.84	1.99	0.44	11	1.81	1.83	0.43	14
3: [1,0,0,2,T]	2.05	2.03	0.26	16	2.08	1.94	0.48	19
4: [1,0,0,-1,T]	2.02	1.98	0.47	13	1.87	1.94	0.32	10
5: [1,0,0,T,0]	2.05	2.02	0.31	23	1.85	1.99	0.46	23
6: [1,0,1,1,T]	1.74	1.85	0.37	15	1.69	1.77	0.38	23
7: [1,0,1,2,T]	1.92	1.95	0.37	16	1.82	1.81	0.33	14
8: [1,0,1,-1,T]	1.86	1.88	0.34	15	1.79	1.87	0.39	22
9: [1,0,1,-2,T]	1.74	1.74	0.43	14	1.82	1.90	0.40	14
10: [1,0,-1,1,T]	2.00	2.00	0.32	22	1.81	1.94	0.42	18
11: [1,0,-2,1,T]	1.97	1.99	0.39	14	2.17	2.14	0.40	18
12: [1,0,-3,1,T]	1.86	1.88	0.34	15	1.79	1.87	0.39	22
13: [1,1,0,T,0]	1.89	1.88	0.31	20	1.82	1.88	0.39	26
14: [1,1,1,1,T]	2.31	2.21	0.41	16	1.75	1.86	0.44	15
15: [1,1,-1,1,T]	2.02	2.01	0.30	11	1.87	1.91	0.32	19
16: [1,1,-2,1,T]	1.95	1.91	0.33	26	1.98	1.97	0.26	18
17: [1,1,-3,1,T]	1.79	1.78	0.25	13	2.00	2.06	0.44	16
18: [1,-2,0,T,0]	1.97	2.05	0.33	24	1.91	1.92	0.44	24
19: [-1,1,0,1,T]	2.11	2.12	0.40	21	1.71	1.88	0.43	17
20: [-1,1,-2,1,T]	1.86	1.92	0.28	23	1.95	1.90	0.36	18
21: [-1,1,-3,1,T]	2.07	2.12	0.57	14	1.81	1.81	0.41	19
All Curves	1.95	1.97	0.37	350	1.85	1.90	0.40	388
Distinct Curves	1.95	1.97	0.37	335	1.85	1.91	0.40	366

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$)
1st Normalized Zero above Central Point

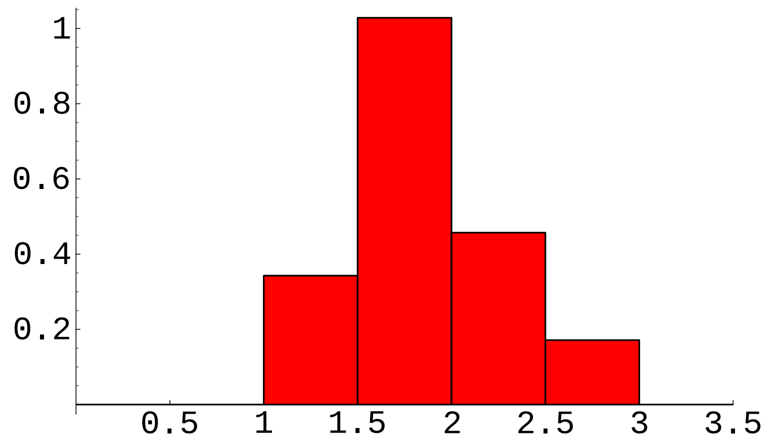

Figure $5a$: 35 curves, $\log(\text{cond}) \in [7.8, 16.1]$, $\widetilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$

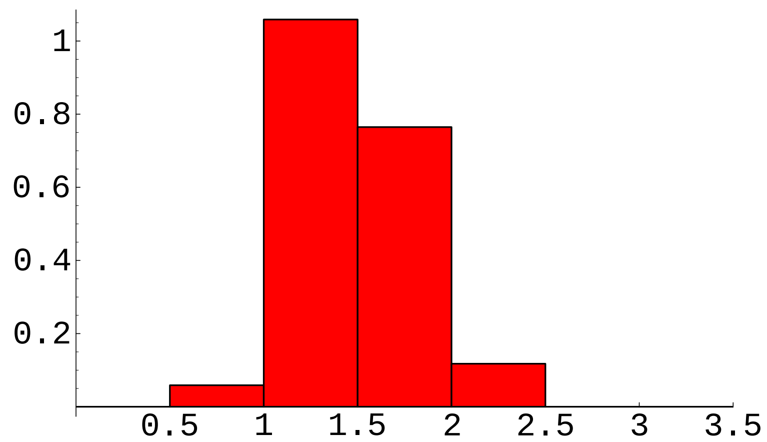

Figure $5b$: 34 curves, $\log(\text{cond}) \in [16.2, 23.3]$, $\widetilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$

Function Field Example (with Sal Butt, Chris Hall)

$y^2 = x^3 + (t^5 + a_1t^4 + a_0)x + (t^3 + b_2t^2 + b_1t + b_0)$, $a_i, b_i \in \mathbb{F}_5$

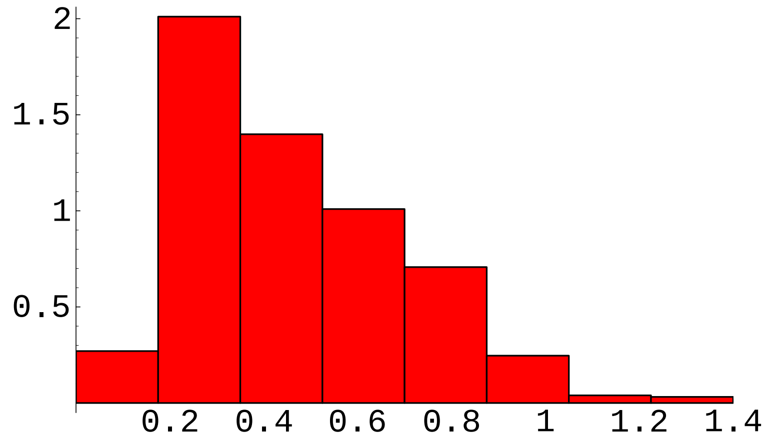

Figure $6a$: Normalized first eigenangle: 719 rank 0 curves.

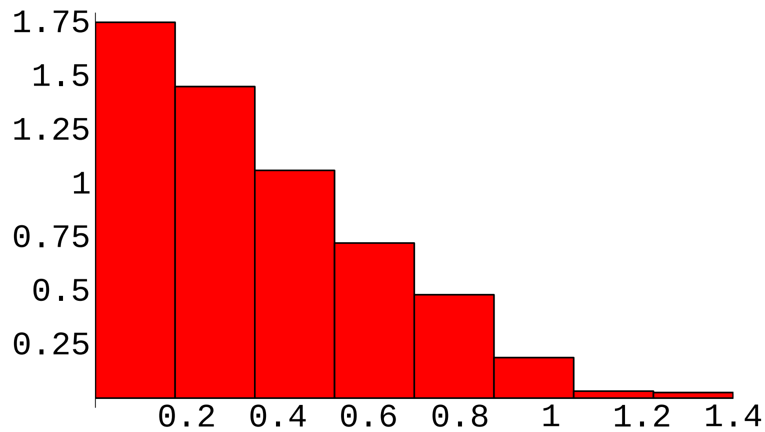

Figure $6b$: Normalized first eigenangle: 978 curves
(719 rank 0 curve, 254 rank 2 curves, 5 rank 4 curves).

Repulsion or Attraction?

Conductors in $[15, 16]$; first set is rank 0 curves from 14 one-parameter families of rank 0 over $\mathbb{Q}$; second set rank 2 curves from 21 one-parameter families of rank 0 over $\mathbb{Q}$. The t-statistics exceed 6.

Family	**2nd vs 1st Zero**	**3rd vs 2nd Zero**	**Number**
Rank 0 Curves	2.16	3.41	863
Rank 2 Curves	1.93	3.27	701

The repulsion from extra zeros at the central point cannot be entirely explained by *only* collapsing the first zero to the central point while leaving the other zeros alone.

Can also interpret as attraction.

Comparison b/w One-Param Families of Different Rank

First normalized zero above the central point.

- The first family is the 701 rank 2 curves from the 21 one-parameter families of rank 0 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$;
- the second family is the 64 rank 2 curves from the 21 one-parameter families of rank 2 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$.

Family	**Median**	**Mean**	**Std. Dev.**	**Number**
Rank 2 Curves (Rank 0 Families)	1.926	1.936	0.388	701
Rank 2 Curves (Rank 2 Families)	1.642	1.610	0.247	64

- t-statistic is 6.60, indicating the means differ.
- The mean of the first normalized zero of rank 2 curves in a family above the central point (for conductors in this range) depends on *how* we choose the curves.

Spacings b/w Normalized Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	**701 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Normalized Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of the j^{th} norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	64 Rank 2 Curves	**23 Rank 4 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2 Curves	**64 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

Modeling the Decrease in Repulsion

- Can the change in zero statistics going from interaction (for small values of the parameter T) to independence (as $T \to \infty$) be modeled using random matrices?
    - Consider orthogonal matrices having a certain number ρ of eigenvalues exactly lying at the central point $+1$.
    - ρ plays the role of a "repulsion parameter" closely related to the rank.

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  - Consider orthogonal matrices having a certain number ρ of eigenvalues exactly lying at the central point $+1$.
  - ρ plays the role of a "repulsion parameter" closely related to the rank.
- The joint PDF of N pairs of eigenvalues $\{e^{i\theta_j}\}_{1 \leq j \leq N}$, taken from random orthogonal matrices having other ρ fixed eigenvalues at $+1$ is

$$d\varepsilon_\rho(\theta_1, \dots, \theta_N) = C_{N,\rho} \prod_{j<k} (\cos\theta_k - \cos\theta_j)^2 \prod_j (1 - \cos\theta_j)^\rho \, d\theta_j.$$

  - This probability measure is well defined for $\rho \in (-\frac{1}{2}, \infty)$.

The Repulsion Parameter ρ

For simplicity, assume that $\mathcal{E}$ is an even orthogonal family depending on a parameter $T \to \infty$.

- The repulsion parameter $\rho = \rho_{\mathcal{E}}(T)$ will monotonically decrease from an initial maximum value $\rho_{\mathcal{E}}(0)$ to a minimum value $\lim_{T\to\infty} \rho_{\mathcal{E}}(T) = 0$ (resp., $\lim_{T\to\infty} \rho_{\mathcal{E}}(T) = 1$ if $\mathcal{E}$ is an odd orthogonal family.)

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- By making ρ vary with T, the statistics of eigenvalues in this model match several of the theoretical and experimental features observed in the critical zeros of $\mathcal{E}$:
  - Repulsion of eigenvalues away from central point when $\rho > 0$. (The larger ρ, the more repulsion.)
  - Independent model statistics when $\rho = 0$.
  - Basically unchanged non-central spacings.

1-Level Density as a Function of ρ

- The standard normalization $x = \frac{N\theta}{\pi}$ makes the eigen-angles θ_j into unit-spaced (on average) “levels” x_j.
- In terms of the x-variable, the limiting 1-level density is given by

$$D_1^{(\rho)}(x) = \rho\delta_0(x) + \pi\left(\frac{\pi x}{2}\left[J_{\rho+\frac{1}{2}}(\pi x)^2 + J_{\rho-\frac{1}{2}}(\pi x)^2\right] - \left(\rho - \tfrac{1}{2}\right) J_{\rho+\frac{1}{2}}(\pi x) J_{\rho-\frac{1}{2}}(\pi x)\right).$$

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

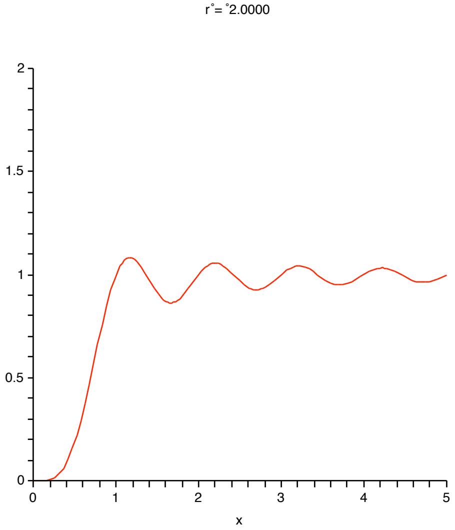

Figure: 1-level density for the ensemble with $\rho = 24/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

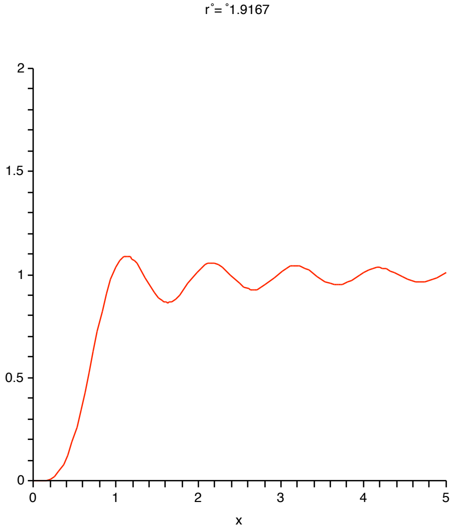

Figure: 1-level density for the ensemble with $\rho = 23/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

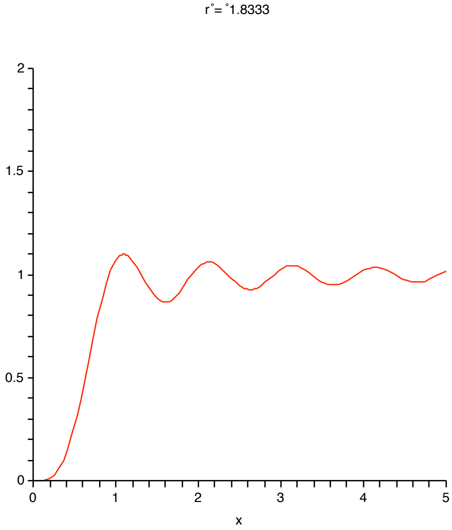

Figure: 1-level density for the ensemble with $\rho = 22/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

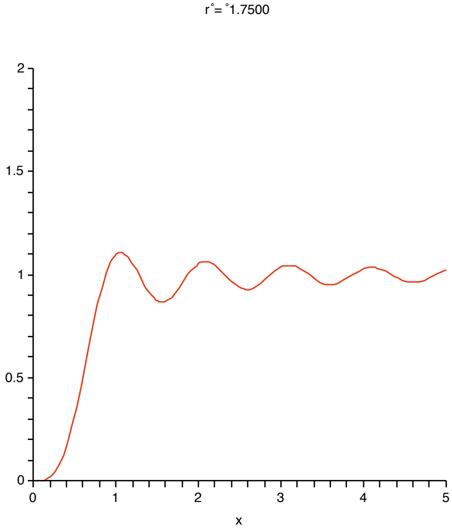

Figure: 1-level density for the ensemble with $\rho = 21/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

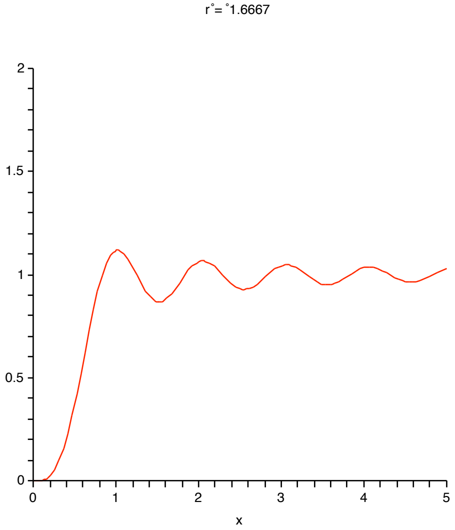

Figure: 1-level density for the ensemble with $\rho = 20/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

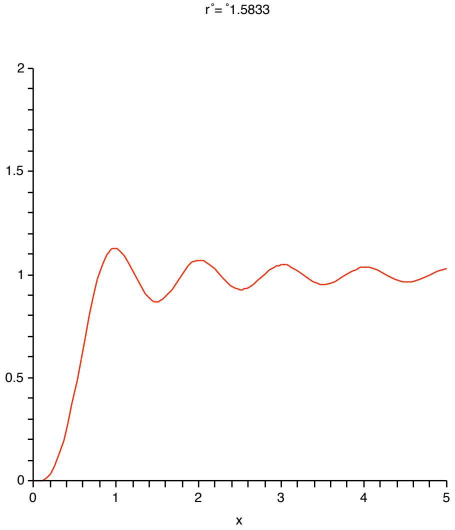

Figure: 1-level density for the ensemble with $\rho = 19/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

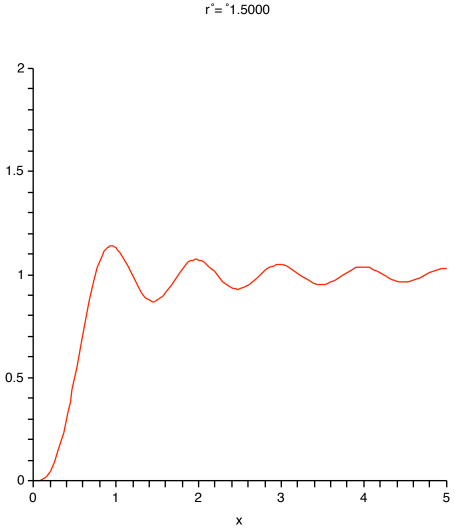

Figure: 1-level density for the ensemble with $\rho = 18/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

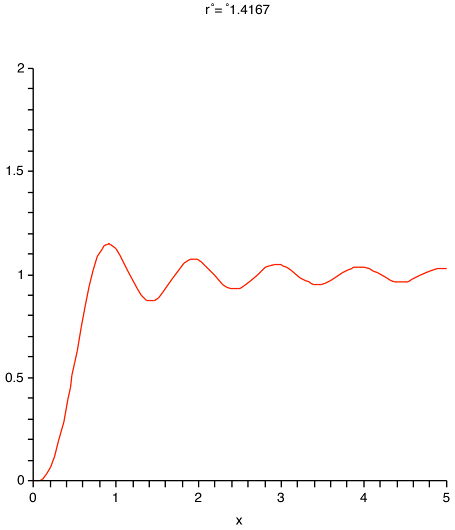

Figure: 1-level density for the ensemble with $\rho = 17/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

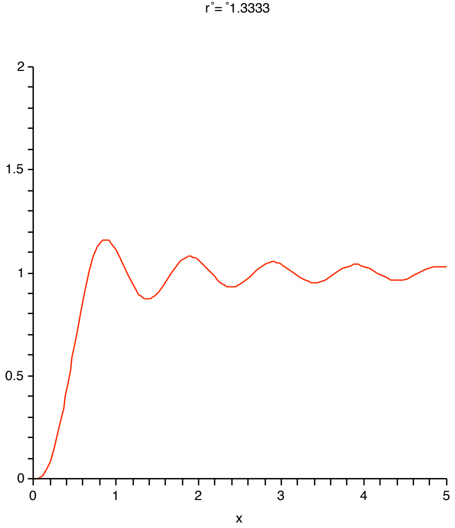

Figure: 1-level density for the ensemble with $\rho = 16/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

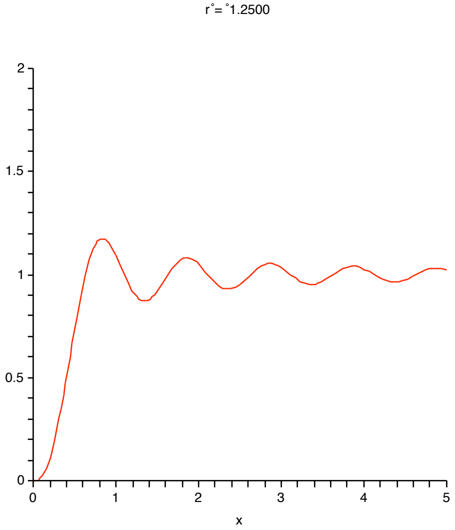

Figure: 1-level density for the ensemble with $\rho = 15/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

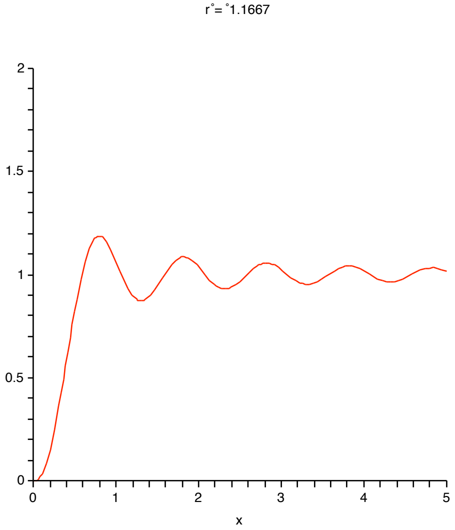

Figure: 1-level density for the ensemble with $\rho = 14/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

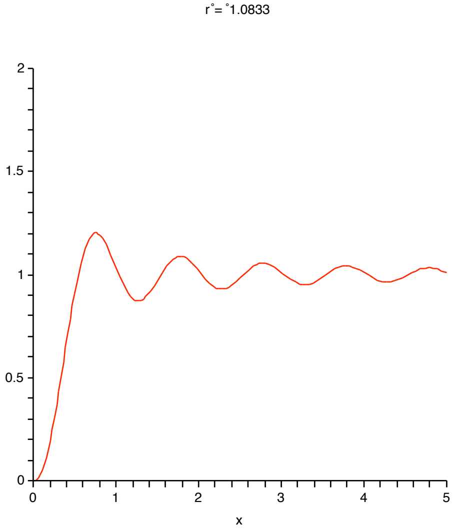

Figure: 1-level density for the ensemble with $\rho = 13/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

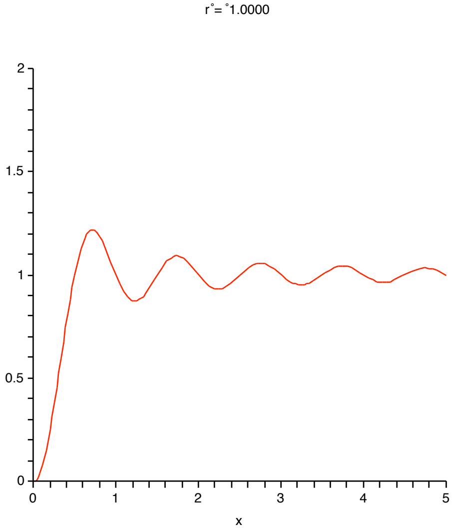

Figure: 1-level density for the ensemble with $\rho = 12/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

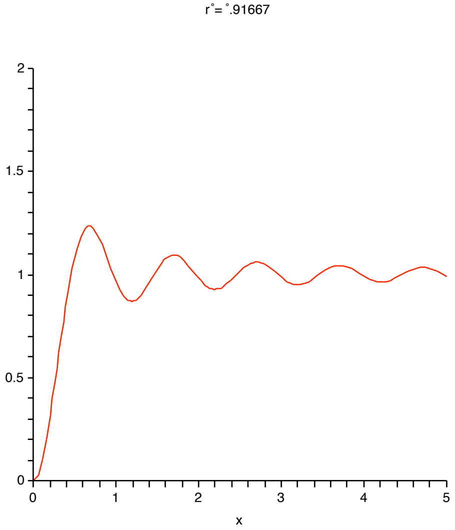

Figure: 1-level density for the ensemble with $\rho = 11/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

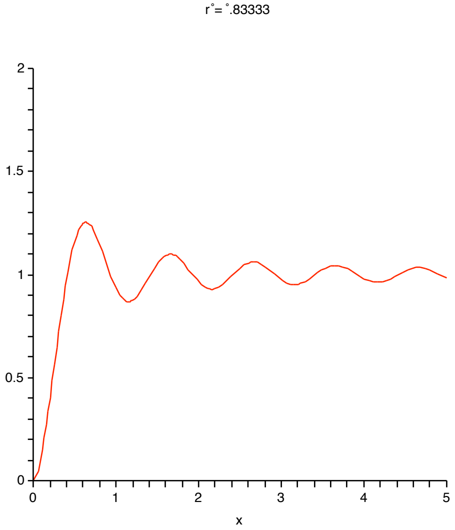

Figure: 1-level density for the ensemble with $\rho = 10/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

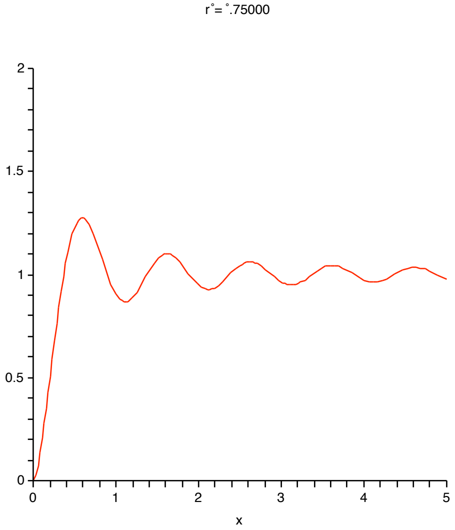

Figure: 1-level density for the ensemble with $\rho = 9/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

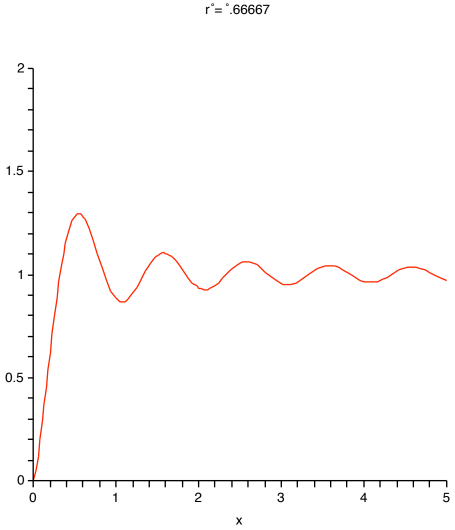

Figure: 1-level density for the ensemble with $\rho = 8/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

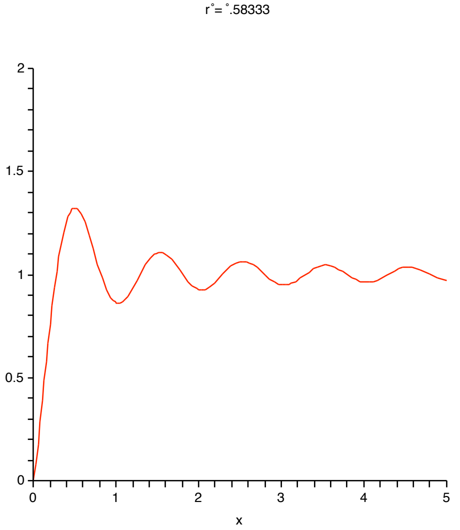

Figure: 1-level density for the ensemble with $\rho = 7/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

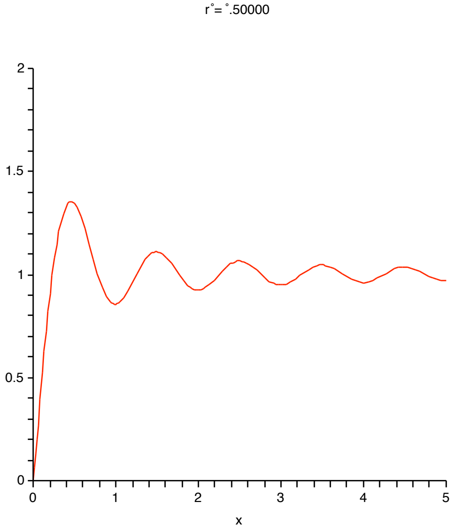

Figure: 1-level density for the ensemble with $\rho = 6/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

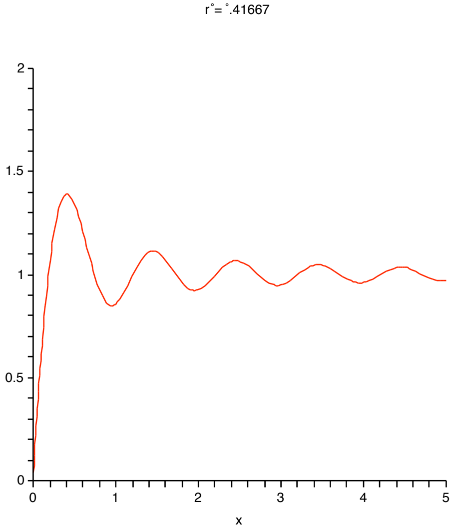

Figure: 1-level density for the ensemble with $\rho = 5/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

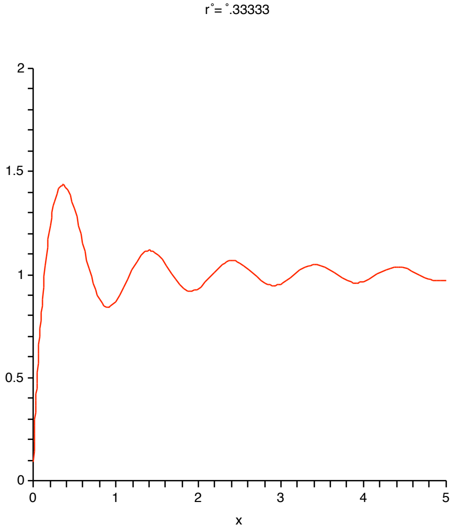

Figure: 1-level density for the ensemble with $\rho = 4/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

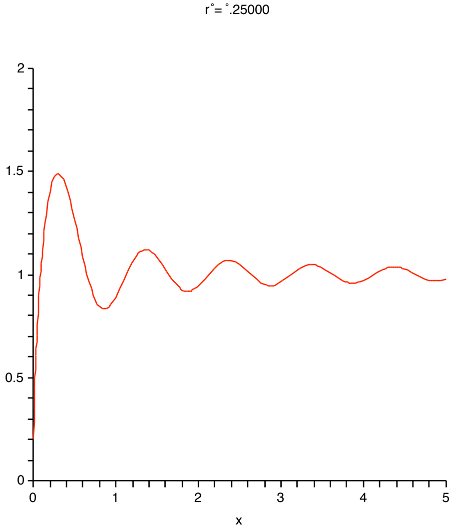

Figure: 1-level density for the ensemble with $\rho = 3/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

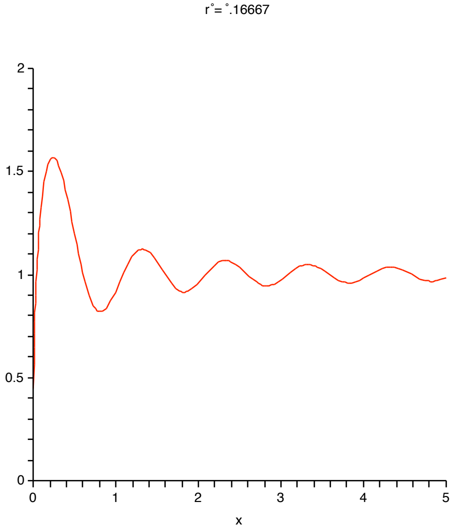

Figure: 1-level density for the ensemble with $\rho = 2/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

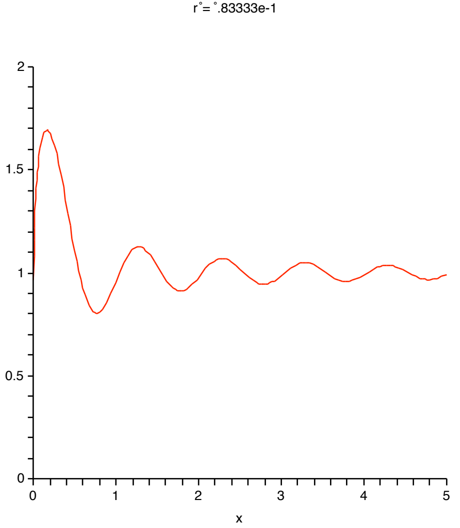

Figure: 1-level density for the ensemble with $\rho = 1/12$.

Animated Example of Decreasing repulsion: $2 \geq \rho \geq 0$

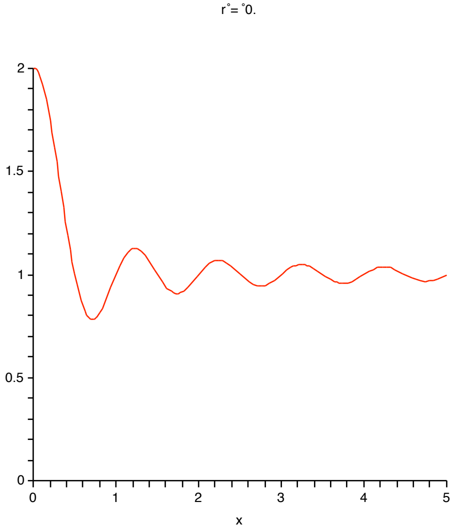

Figure: 1-level density for the ensemble with $\rho = 0/12$.

The Effect of the Parameter ρ

- As ρ varies from $\rho(0)$ to 0 the "central repulsion" decreases and, at $r = 0$, it disappears completely.
- Any $\rho > 0$ merely tends to shift all the eigenvalues to the right: they are pushed away, but the relative spacings between them are basically unchanged.

What is the Function $\rho_{\mathcal{E}}(T)$?

- First issue: What should $\rho_{\mathcal{E}}(0)$ be?
  - Choice #1: Take $\rho_{\mathcal{E}}(0)$ equal to the geometric rank of a family $\mathcal{E}$ over $\mathbb{Q}(T)$.

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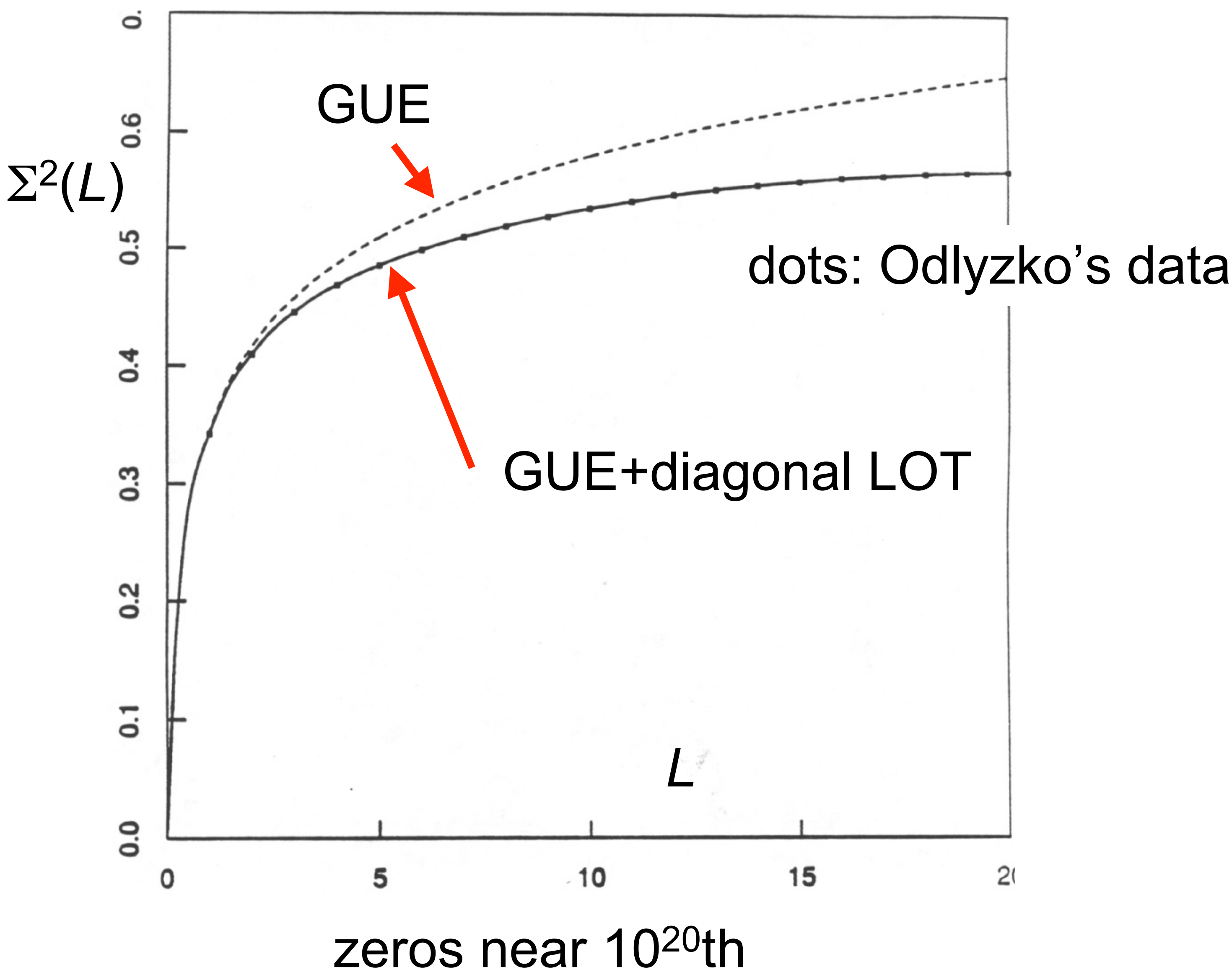
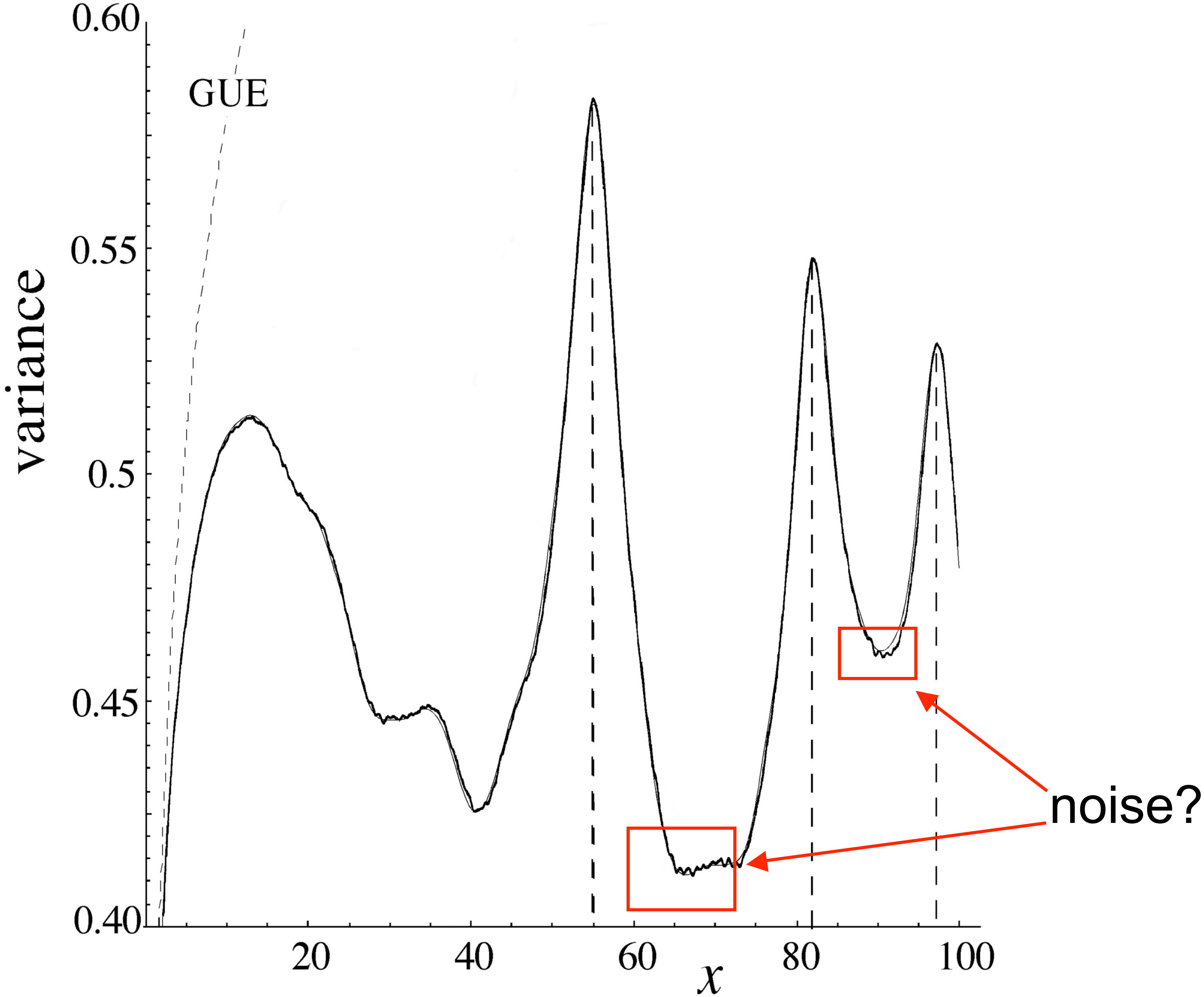
- Second issue: How does $\rho_{\mathcal{E}}(T)$ vary with T?
  - r should probably go to zero inversely with the log conductors of curves in $\mathcal{E}$.
    - Best bet so far:

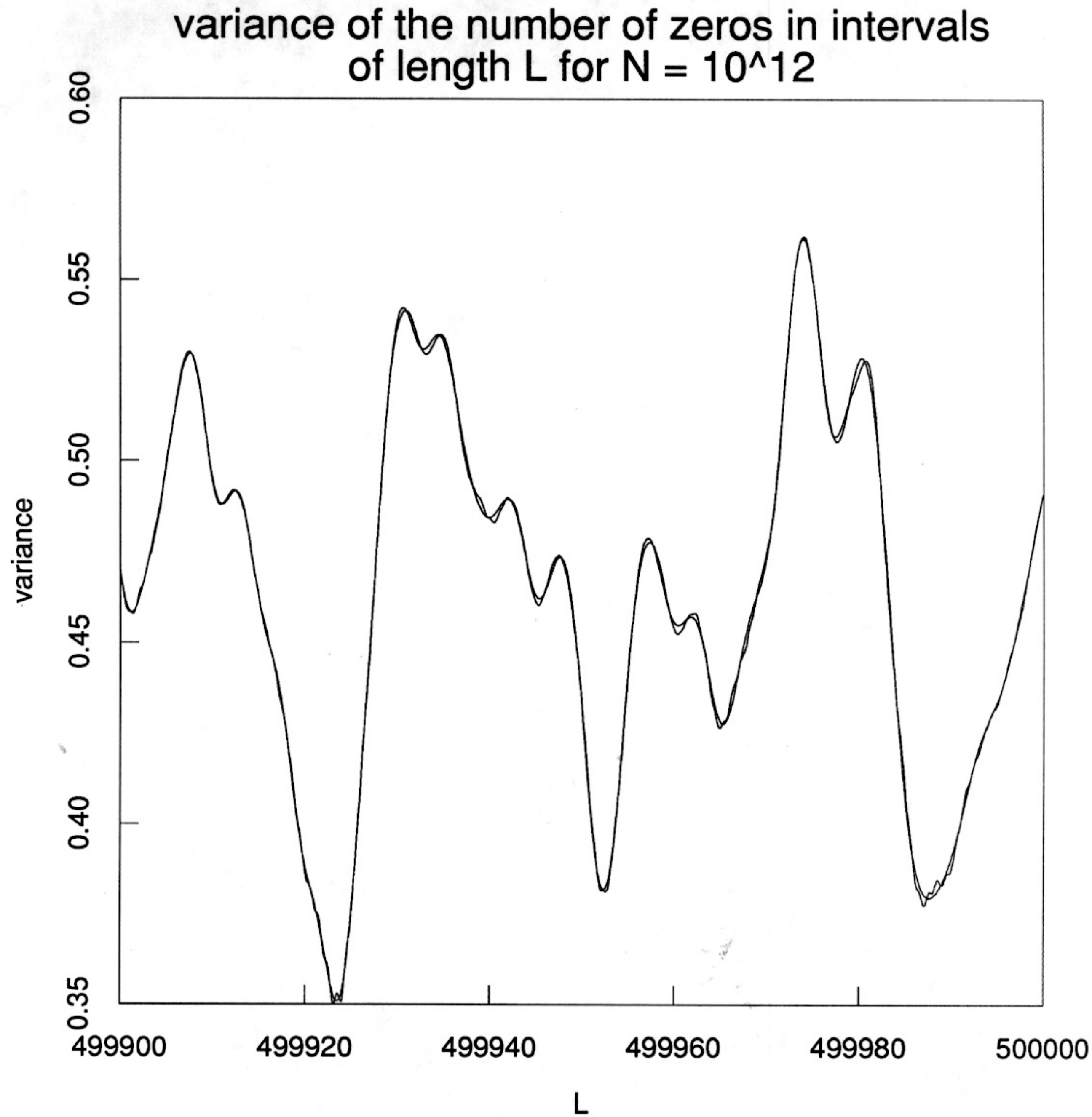
$$\rho_{\mathcal{E}}(T) = \frac{\langle r \rangle_{\mathcal{E}(T)}}{\log T} \qquad (+1 \text{ if odd family.})$$

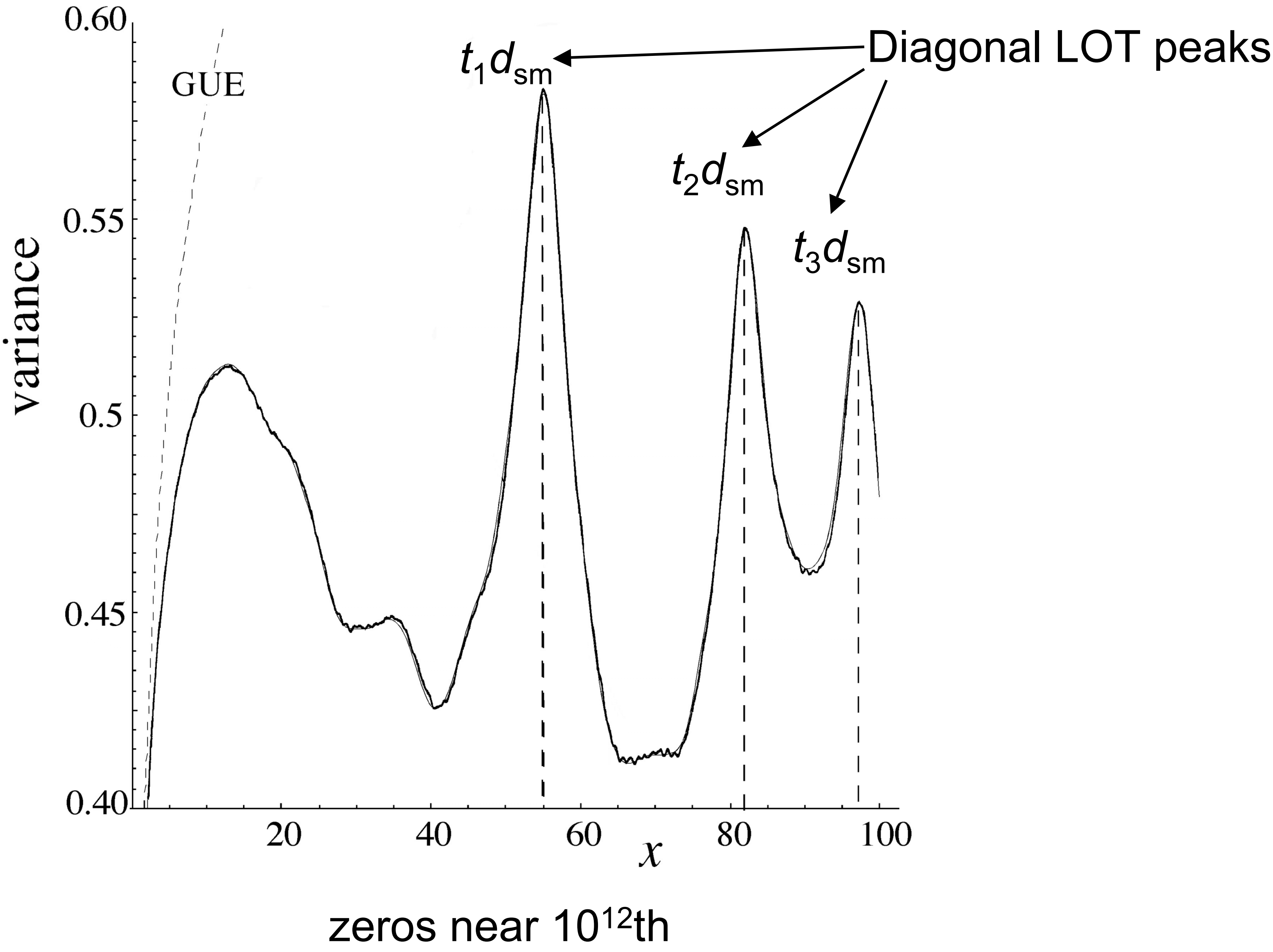
deviations from random matrix theory require contributions from ***lower order terms (LOTs)***, which are non-universal

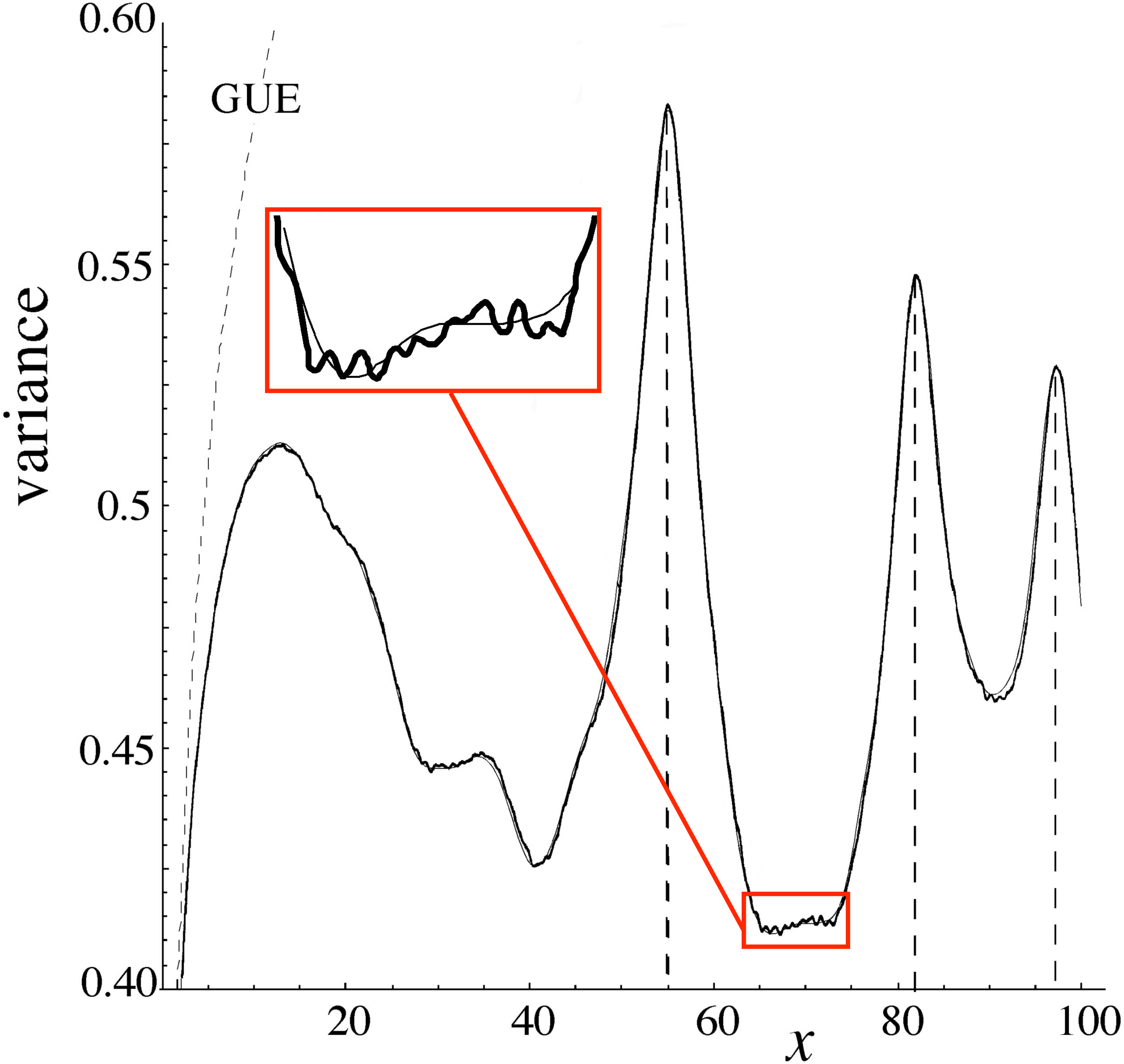




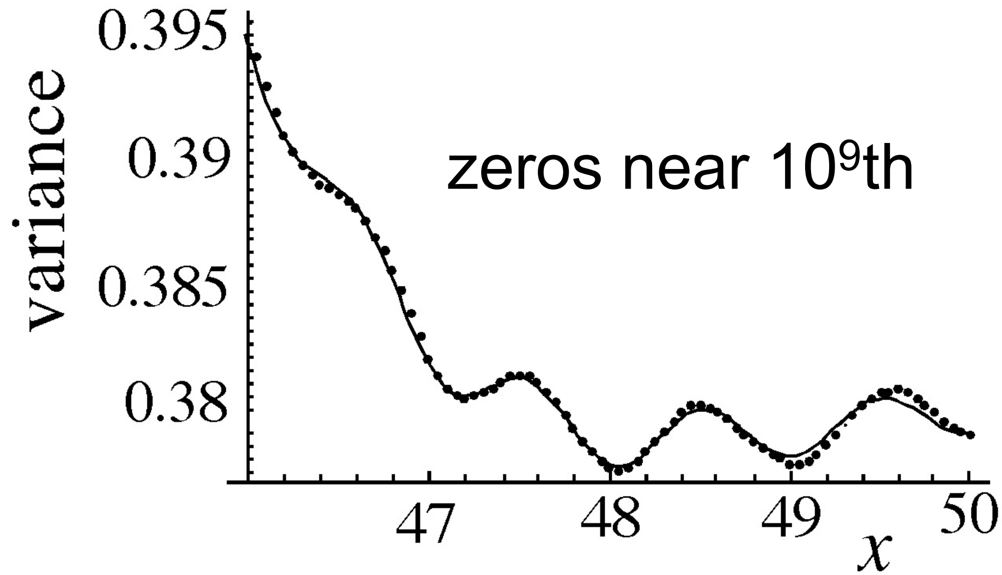
zeros near 10^{12}th


variance of the number of zeros in intervals
of length L for N = 10^12
variance
L
0.35
0.40
0.45
0.50
0.55
0.60
499900
499920
499940
499960
499980
500000


Diagonal LOT peaks
$t_1 d_{sm}$
$t_2 d_{sm}$
$t_3 d_{sm}$
GUE
variance
0.60
0.55
0.5
0.45
0.40
20
40
60
80
100
x
zeros near 10^{12}th



0.60
0.55
0.5
0.45
0.40
variance
GUE
20
40
60
80
100
x



R_{GUE}+ *diagonal LOT* + *off-diagonal LOT*

- Katz/Sarnak philosophy:
  In the limit as the conductor tends to ∞

  zero statistics for families of L-functions
  "="
  eigenvalue statistics of one of the classical compact groups.

- However for finite conductor we observe repulsion of low lying zeros from the critical point. This behaviour is not captured in the models coming from the classical compact groups. We believe that a study of lower order terms will help to understand this phenomenon.

- We investigate the family quadratic twists coming from an elliptic curve L-function (and for simplicity we consider those with even functional equation) because
  - Experimental data from Mike Rubinstein's lcalc.
  - Lower order terms for this family from the ratios conjectures.

  So we can compare theory and data.

- The ratios conjectures (Conrey, Farmer, Zirnbauer) give good estimates for quantities like

$$\sum_{0<d\leq X} \frac{\prod_{k=1}^{K} L(1/2+\alpha_k,\chi_d)}{\prod_{q=1}^{Q} L(1/2+\gamma_q,\chi_d)}.$$

  Simpliest case when $K = Q = 1$.

- For our family we are interested in the following ratio with $\Re(\alpha), \Re(\gamma) > 0$

$$\sum_{d \le X} \frac{L_E(1/2+\alpha, \chi_d)}{L_E(1/2+\gamma, \chi_d)} =: R_E(\alpha, \gamma)$$

and observe that

$$\sum_{d \le X} \frac{L'_E(1/2+r, \chi_d)}{L_E(1/2+r, \chi_d)} = \frac{d}{d\alpha} R_E(\alpha, \gamma)\Big|_{\alpha=\gamma=r}.$$

- For the rest of the talk we focus on the 1-level-density:

$$D_1(\varphi) = \frac{1}{X^*} \sum_{d \le X} \sum_{\gamma_d} \varphi(\gamma_d)$$

where φ is a suitable test function, say an even Schwartz-function and γ_d the ordinate of a generic zero of $L_E(s, \chi_d)$ on the critical line.

By the argument principle we can write

$$D_1(\varphi) = \frac{1}{X^*} \sum_{d \le X} \frac{1}{2\pi i} \left(\int_{(c)} - \int_{(1-c)} \right) \frac{L'(s,\chi_d)}{L(s,\chi_d)} \varphi(-i(s-1/2))ds$$

where $3/4 > c > 1/2 + 1/\log X$.

Hence: If we have a conjecture for

$$\sum_{d \le X} \frac{L_E'(s,\chi_d)}{L_E(s,\chi_d)} \tag{1}$$

we can also give a conjectural answer for $D_1(\varphi)$.

Using the ratios conjectures we get an estimate for (1).

- From the ratios conjecture we get for the 1-level-density

$$D_1(\varphi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(t) \frac{1}{X^*} \sum_{d \le X} \left(2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right)\right.$$

$$+ \frac{\Gamma'}{\Gamma}(1/2+it) + \frac{\Gamma'}{\Gamma}(1/2-it)$$

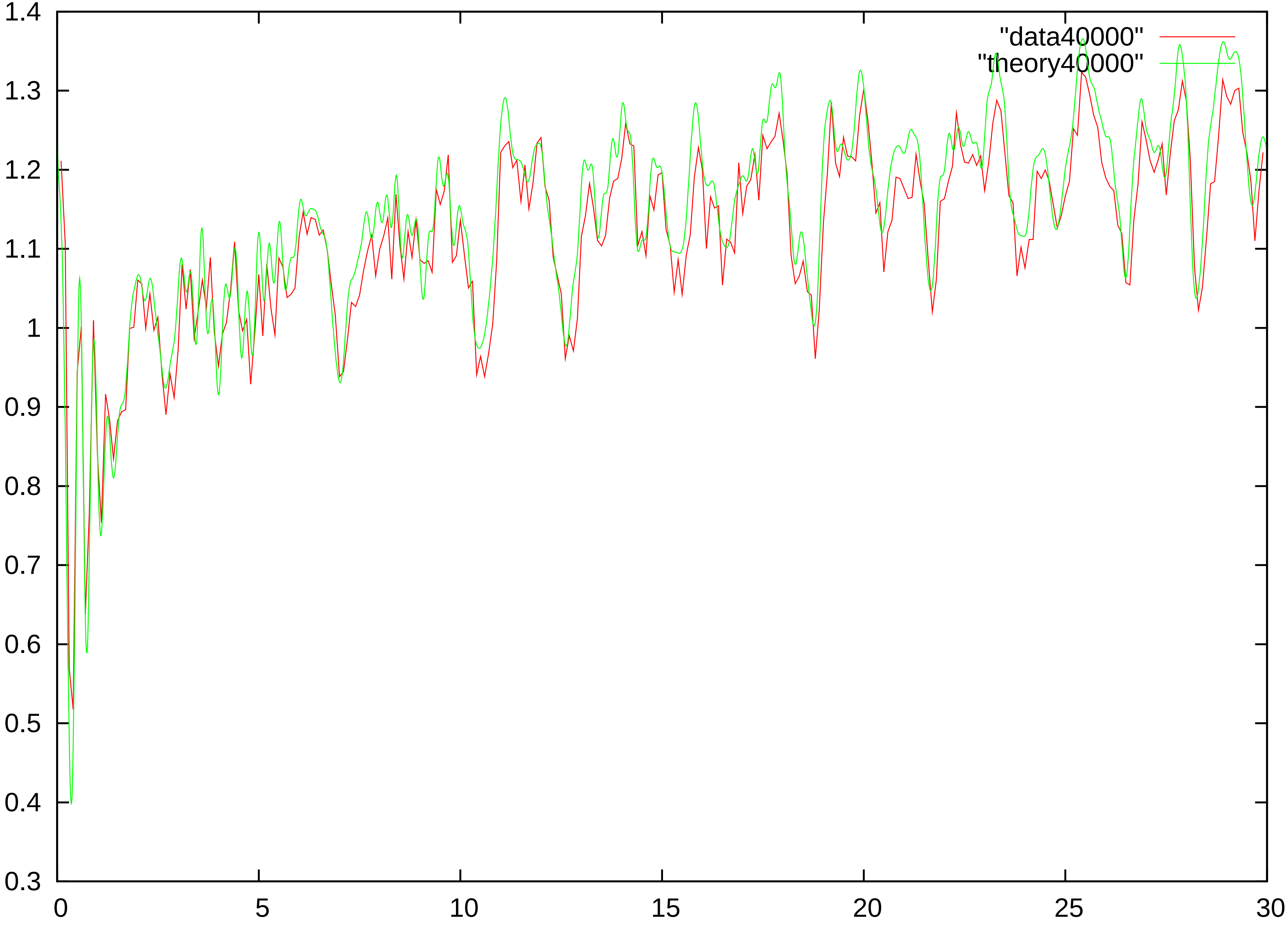
$$+ 2\Big[-\frac{\zeta'(1+2it)}{\zeta(1+2it)} + \frac{L_E'(\mathrm{sym}^2, 1+2it)}{L_E(\mathrm{sym}^2, 1+2it)} + A_E'(it, it)$$

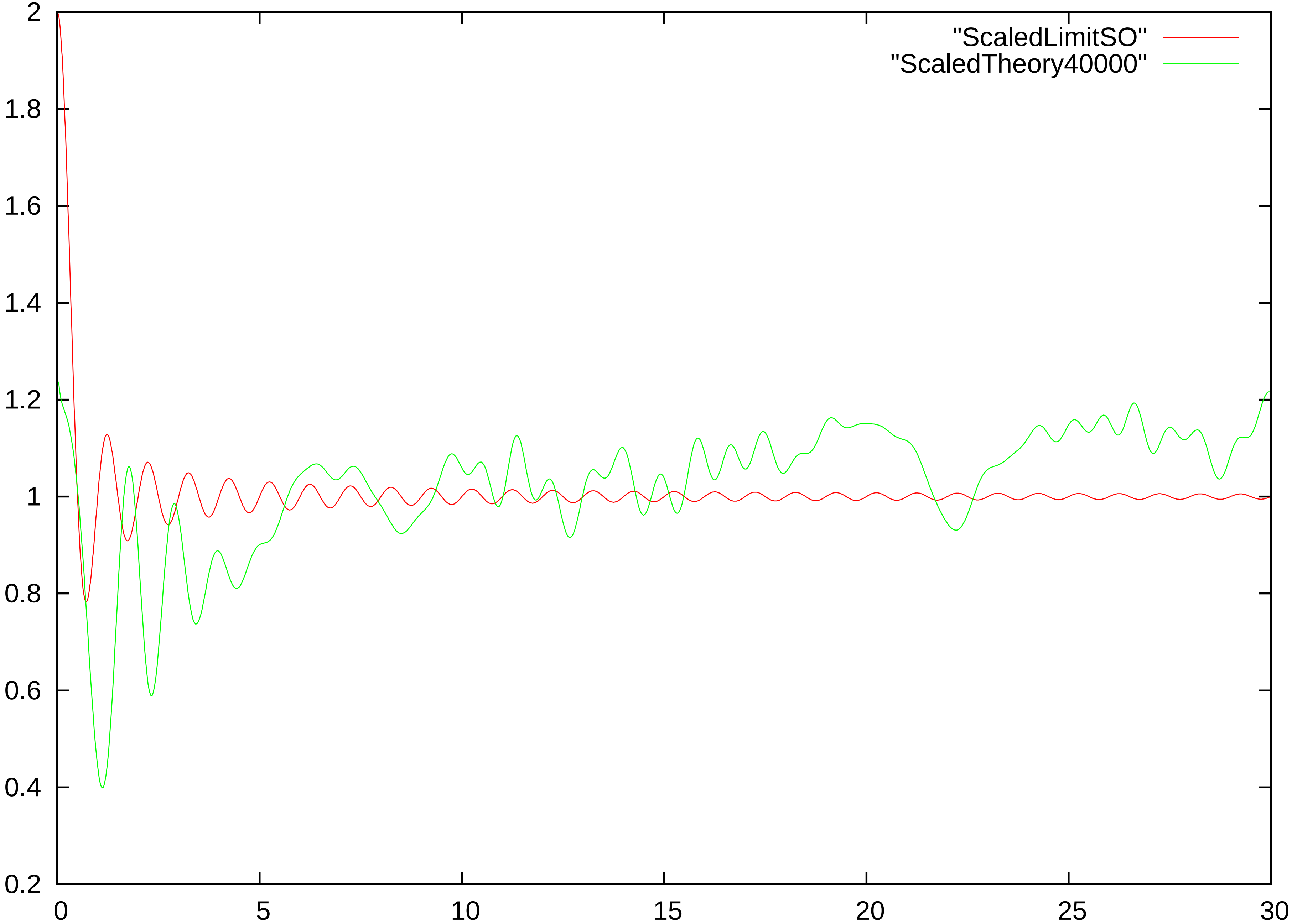
$$- \left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2it} \frac{\Gamma(1/2-it)}{\Gamma(1/2+it)} \frac{\zeta(1+2it) L_E(\mathrm{sym}^2, 1-2it)}{L_E(\mathrm{sym}^2, 1)} A_E(-it, it)\Big]$$

$$+ O(X^{-1/2+\varepsilon})$$

where M is the conductor of the elliptic curve E and A_E is a product over primes. We note that

– the ratios conjecture give all terms down to $O(X^{-1/2+\varepsilon})$ which is a very precise prediction.


"data40000"
"theory40000"
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0
5
10
15
20
25
30


"ScaledLimitSO"
"ScaledTheory40000"
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1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
5
10
15
20
25
30

Appendices

The first appendix list various standard conjectures. The second appendix gives the formula to numerically approximate the analytic rank of an elliptic curve. For a curve of conductor C_E, one needs about $\sqrt{C_E}\log C_E$ Fourier coefficients. The third is the statement (with assumptions) of the main theoretical result for the one-level density of one-parameter families of Elliptic curves over $\mathbb{Q}(T)$.

Appendix I: Standard Conjectures

Generalized Riemann Hypothesis (for Elliptic Curves) *Let $L(s, E)$ be the (normalized) L-function of the elliptic curve E. Then the non-trivial zeros of $L(s, E)$ satisfy Re$(s) = \frac{1}{2}$.*

Birch and Swinnerton-Dyer Conjecture [BSD1], [BSD2] *Let E be an elliptic curve of geometric rank r over $\mathbb{Q}$ (the Mordell-Weil group is $\mathbb{Z}^r \oplus T$, T is the subset of torsion points). Then the analytic rank (the order of vanishing of the L-function at the central point) is also r.*

Tate's Conjecture for Elliptic Surfaces [Ta] *Let $\mathcal{E}/\mathbb{Q}$ be an elliptic surface and $L_2(\mathcal{E}, s)$ be the L-series attached to $H^2_{\acute{e}t}(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$. Then $L_2(\mathcal{E}, s)$ has a meromorphic continuation to* **C** *and satisfies $-$ord$_{s=2}L_2(\mathcal{E}, s) =$ rank $NS(\mathcal{E}/\mathbb{Q})$, where $NS(\mathcal{E}/\mathbb{Q})$ is the $\mathbb{Q}$-rational part of the Néron-Severi group of $\mathcal{E}$. Further, $L_2(\mathcal{E}, s)$ does not vanish on the line Re$(s) = 2$.*

Most of the 1-param families we investigate are rational surfaces, where Tate's conjecture is known. See [RSi].

Appendix II: Numerically Approximating Ranks: Preliminaries

Cusp form f, level N, weight 2:

$$\begin{aligned} f(-1/Nz) &= -\epsilon N z^2 f(z) \\ f(i/y\sqrt{N}) &= \epsilon y^2 f(iy/\sqrt{N}). \end{aligned}$$

Define

$$\begin{aligned} L(f,s) &= (2\pi)^s \Gamma(s)^{-1} \int_0^{i\infty} (-iz)^s f(z) \frac{dz}{z} \\ \Lambda(f,s) &= (2\pi)^{-s} N^{s/2} \Gamma(s) L(f,s) = \int_0^{\infty} f(iy/\sqrt{N}) y^{s-1} dy. \end{aligned}$$

Get

$$\Lambda(f,s) = \epsilon \Lambda(f, 2-s), \quad \epsilon = \pm 1.$$

To each E corresponds an f, write $\int_0^\infty = \int_0^1 + \int_1^\infty$ and use transformations.

Algorithm for $L^r(s,E)$: I

$$\begin{aligned}\Lambda(E,s) &= \int_0^\infty f(iy/\sqrt{N})y^{s-1}dy \\ &= \int_0^1 f(iy/\sqrt{N})y^{s-1}dy + \int_1^\infty f(iy/\sqrt{N})y^{s-1}dy \\ &= \int_1^\infty f(iy/\sqrt{N})(y^{s-1}+\epsilon y^{1-s})dy.\end{aligned}$$

Differentiate k times with respect to s:

$$\Lambda^{(k)}(E,s) = \int_1^\infty f(iy/\sqrt{N})(\log y)^k(y^{s-1}+\epsilon(-1)^k y^{1-s})dy.$$

At $s=1$,

$$\Lambda^{(k)}(E,1) = (1+\epsilon(-1)^k)\int_1^\infty f(iy/\sqrt{N})(\log\ y)^k dy.$$

Trivially zero for half of k; let r be analytic rank.

Algorithm for $L^r(s, E)$: II

$$\begin{aligned}\Lambda^{(r)}(E,1) &= 2\int_1^\infty f(iy/\sqrt{N})(\log\ y)^r dy \\ &= 2\sum_{n=1}^\infty a_n \int_1^\infty e^{-2\pi n y/\sqrt{N}}(\log\ y)^r dy.\end{aligned}$$

Integrating by parts

$$\Lambda^{(r)}(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^\infty \frac{a_n}{n}\int_1^\infty e^{-2\pi n y/\sqrt{N}}(\log\ y)^{r-1}\frac{dy}{y}.$$

We obtain

$$L^{(r)}(E,1) = 2r!\sum_{n=1}^\infty \frac{a_n}{n} G_r\left(\frac{2\pi n}{\sqrt{N}}\right),$$

where

$$G_r(x) = \frac{1}{(r-1)!}\int_1^\infty e^{-xy}(\log y)^{r-1}\frac{dy}{y}.$$

Expansion of $G_r(x)$

$$G_r(x) = P_r\left(\log\frac{1}{x}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-r}}{n^r \cdot n!} x^n$$

$P_r(t)$ is a polynomial of degree r, $P_r(t) = Q_r(t-\gamma)$.

$$\begin{aligned}
Q_1(t) &= t;\\
Q_2(t) &= \frac{1}{2}t^2 + \frac{\pi^2}{12};\\
Q_3(t) &= \frac{1}{6}t^3 + \frac{\pi^2}{12}t - \frac{\zeta(3)}{3};\\
Q_4(t) &= \frac{1}{24}t^4 + \frac{\pi^2}{24}t^2 - \frac{\zeta(3)}{3}t + \frac{\pi^4}{160};\\
Q_5(t) &= \frac{1}{120}t^5 + \frac{\pi^2}{72}t^3 - \frac{\zeta(3)}{6}t^2 + \frac{\pi^4}{160}t - \frac{\zeta(5)}{5} - \frac{\zeta(3)\pi^2}{36}.
\end{aligned}$$

For $r = 0$,

$$\Lambda(E,1) = \frac{\sqrt{N}}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{n}\, e^{-2\pi n y/\sqrt{N}}.$$

Need about $\sqrt{N}$ or $\sqrt{N}\log N$ terms.

Appendix III: 1-Level Density

Definitions:

$$D_{n,\mathcal{F}}(\varphi) = \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{\substack{j_1,\dots,j_n \\ j_i\neq\pm j_k}}\prod_i \varphi_i\left(\frac{\log C_E}{2\pi}\gamma_E^{(j_i)}\right)$$

$D^{(r)}_{n,\mathcal{F}}(\varphi)$: n-level density with contribution of r zeros at central point removed.

$\mathcal{F}_N$: Rational one-parameter family,
$t \in [N, 2N]$, conductors monotone.

ASSUMPTIONS

1-parameter family of Ell Curves, rank r over $\mathbb{Q}(T)$, rational surface. Assume

- GRH;
- $j(t)$ non-constant;
- Sq-Free Sieve if $\Delta(t)$ has irr poly factor of deg ≥ 4.

Pass to positive percent sub-seq where conductors polynomial of degree m.

φ_i even Schwartz, support σ_i:

- $\sigma_1 < \min\left(\frac{1}{2}, \frac{2}{3m}\right)$ for 1-level
- $\sigma_1 + \sigma_2 < \frac{1}{3m}$ for 2-level.

MAIN RESULT

Theorem (Miller 2004): Under previous conditions, as $N \to \infty$, $n = 1, 2$:

$$D^{(r)}_{n,\mathcal{F}_N}(\varphi) \longrightarrow \int \varphi(x) W_{\mathcal{G}}(x) dx,$$

where

$$\mathcal{G} \ = \ \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

1 and 2-level densities confirm Katz-Sarnak, B-SD predictions for small support.

Examples

Constant-Sign Families:

1. $y^2 = x^3 + 2^4(-3)^3(9t+1)^2$,
   $9t+1$ Square-Free: all even.

2. $y^2 = x^3 \pm 4(4t+2)x$,
   $4t+2$ Square-Free:
   $+$ all odd, $-$ all even.

3. $y^2 = x^3 + tx^2 - (t+3)x + 1$,
   t^2+3t+9 Square-Free: all odd.

First two rank 0 over $\mathbb{Q}(T)$, third is rank 1.

Without 2-Level Density, couldn't say *which* orthogonal group.

Examples (cont)

Rational Surface of Rank 6 over $\mathbf{Q}(t)$:

$$\begin{aligned} y^2 \ &= \ x^3 + (2at - B)x^2 + (2bt - C)(t^2 + 2t - A + 1)x \\ &\quad + (2ct - D)(t^2 + 2t - A + 1)^2 \end{aligned}$$

$$\begin{aligned} A &= 8,916,100,448,256,000,000 \\ B &= -811,365,140,824,616,222,208 \\ C &= 26,497,490,347,321,493,520,384 \\ D &= -343,107,594,345,448,813,363,200 \\ a &= 16,660,111,104 \\ b &= -1,603,174,809,600 \\ c &= 2,149,908,480,000 \end{aligned}$$

Need GRH, Sq-Free Sieve to handle sieving.

Appendix IV: t-Statistics

The Pooled Two-Sample t-Procedure is

$$t \;=\; (\overline{X_1} - \overline{X_2}) \Big/ s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}},$$

where $\overline{X_i}$ is the sample mean of n_i observations of population i, s_i is the sample standard deviation and

$$s_p \;=\; \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

is the pooled variance; t has a t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

The Unpooled Two-Sample t-Procedure is

$$t \;=\; (\overline{X_1} - \overline{X_2}) \Big/ \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}};$$

this is approximately a t distribution with

$$\frac{(n_1 - 1)\,(n_2 - 1)\,(n_2 s_1^2 + n_1 s_2^2)^2}{(n_2 - 1)n_2^2 s_1^4 + (n_1 - 1)n_1^2 s_2^4}$$

degrees of freedom

Bibliography Caveat

Warning: this bibliography hasn't been updated for a few years, and could be a little out of date. It is meant to serve as a first reference. If you email sjmiller@math.brown.edu with additional, relevant references, they will be added.

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