Rational curves on algebraic varieties

The American Institute of Mathematics

The following compilation of participant contributions is only intended as a lead-in to the ARCC workshop "Rational curves on algebraic varieties ." This material is not for public distribution.

Corrections and new material are welcomed and can be sent to `workshops@aimath.org`

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Table of Contents

Chapter A: Participant Contributions

A.1 Chen, Yifei

An open problem.

Let S be a complete rational surface with rational singularities over $\mathbb{C}$. Is there a rational curve on S not passing through singularities and passing through two generic points?

A.2 Colliot-Thélène, Jean-Louis

I am most interested in the arithmetic of varieties which are close to being (geometrically) rational. The original motivation lies in the study of rational points over the rationals.

For the time being, we see serious developments on the complex algebraic geometry side: rationally connected varieties have been around for quite a while, these days we have 1-rationally connected varieties – still in development – and there are applications to the existence and the density of rational points when the base field is a function field in one or two variables over an algebraically closed field.

We also see that conditions on Chow groups of zero-cycles, which are implied by rational connectedness, play a rôle in the study of rational points of varieties over finite fields – here étale cohomology, as opposed to coherent cohomology, is the tool.

A result on one side often suggests one on the other side (example : Chevalley–Warning versus Tsen) but we see essentially no case when one can use one side to get results on the other side. I find this baffling.

To mention some slightly more precise points:

I shall be interested in discussing the precise Galois cohomological condition to be imposed on (the yet to be defined) 1-rational connected varieties over a function field in two variables over the complexes to guarantee that they have a rational point.

I will also ask participants which kind of application might have my remark (June 2006) that Kollár's recent solution of Ax's conjecture (char. zero) implies, for $n \geq d^2$, a nice property of special fibres of one-parameter families of hypersurfaces of degree d in $\mathbb{P}^n$.

A.3 de Jong, Aise

I would like to describe joint work with Jason Starr on higher rational connectedness and (unpublished) ideas of Phillipe Gille relating this to Galois cohomology of simply connected simple linear algebraic groups over function fields of surfaces.

A.4 Esnault, Hélène

I'll write more closer to the workshop: subjects are

I- congruences for the number of rational points on rationally connected varieties over finite fields:

* smooth projective

* singular projective

* reduction mod p of a smooth projective variety over a local field

level: why we can't go up so far

natural questions on the above.

II- over a field of finite type over the rationals, in particular over number fields: groupoid scheme of smooth varieties, and its relation to rational points.

A.5 Hassett, Brendan

I am interested in applications of rationally connected varieties and the deformation-theoretic techniques to problems from arithmetic geometry. Examples include:

A. weak approximation for rationally connected varieties over function fields of curves;
B. density results for rational points of K3 surfaces over function fields of curves;
C. approximation and density results for integral points of log Fano varieties over function fields of curves.

I would like to learn more about progress in other arithmetic contexts:

A. rational points of rationally-connected varieties over finite and local fields;
B. rationally simply-connected varieties, especially over function fields of surfaces.

A.6 Hwang, Jun-Muk

Open problems:

1. Is there an n-dimensional Fano manifold of Picard number 1 embedded in a projective space which contains a linear subspace of dimension k with trivial normal bundle? There are lots of examples with $k = 1$, e.g. cubic threefolds. It is easy to see that there is no such linear subspace if $2k > n$. So the interesting case is when $2 < 2k \leq n$.

2. Let X be a Fano manifold of Picard number 1 covered by an irreducible family M of rational curves of degree 2 with respect to $-K_X$. Let U_1, U_2 be two connected open subsets of X in classical topology and $\varphi : U_1 \to U_2$ be a biholomorphic map. Assume that for each member C of M, there exists another member C' of M such that $(C \cap U_1) = C' \cap U_2$. Can we extend φ to a biregular automorphism of X? This is an anaolg of Liouville theorem in conformal geometry. This has been checked for a few cases: hypersurfaces of degree n in $n + 1$-dimensional projective space, Mukai-Umemura threefolds.

A.7 Kebekus, Stefan

Positivity of the tangent bundle, and rationally connected foliations.

Miyaoka's classical uniruledness criterion asserts that a variety is uniruled, if the restriction of the tangent bundle to a general complete intersection curve contains a positive subbundle. In the proof, Miyaoka observed that the associated maximally destabilizing subsheaf yields a foliation. He then employs his theory of "deformation along a foliation" in order to produce rational curve by bend-and-break. In a joint paper with Luis Solá and Matei Toma ("Rationally connected foliations after Bogomolov and McQuillan", JAG, 2007), we have studied the relation between positivity of the tangent bundle and existence of rational curves in more detail. Following ideas of Bogomolov-McQuillan, we show that a foliation which is positive along an arbitrary curve will always have algebraic and rationally connected leaves. Apart from a vanishing theorem for vector bundles in positive characteristic, the proof employs only standard techniques of Mori theory and does not make any reference to the more involved properties of foliations in characteristic p.

In the workshop I would like to pose the question if there is a Miyaoka-type criterion for a larger class of curves than general complete intersections —since "uniruledness" is

a birational property, it seems reasonable to look for a class of curves whose definition is birational, or does at least not depend on the choice of a polarization.

Extension of logarithmic forms over exceptional sets, and varieties mapping to the moduli stack of varieties of general type.

Shafarevich's classical hyperbolicity conjecture asserts that a family of curves over a quasi-projective 1-dimensional base is isotrivial unless the logarithmic Kodaira dimension of the base is positive. More generally it has been conjectured by Viehweg that the base of a smooth family of canonically polarized varieties is of log general type if the family is of maximal variation. In a recent work with Sándor Kovács ("Families of canonically polarized varieties over surfaces", math.AG/0511378v2 to appear in Invent. Math.), we relate the variation of a family to the logarithmic Kodaira dimension of the base and give an affirmative answer to Viehweg's conjecture for families over quasi-projective surfaces.

Trying to generalize these results to higher dimensional base manifolds, one is led to consider extension problems for logarithmic pluri-forms. For a simple example, let X be a surface, $E \subset X$ a (-1)-curve and $\omega \in H^0\bigl(X, \Omega^p_X(*E)\bigr)$ a p-form that is allowed to have arbitrary poles along E. An elementary computation shows that ω is in fact everywhere regular on X, i.e. $\omega \in H^0\bigl(X, \Omega^p_X\bigr)$. It is not very difficult to show that similar results hold for sections in the symmetric product of the sheaf logarithmic differentials, if E contracts to a log terminal surface singularity. I would like to ask if similar results hold in higher dimensions. It seems likely that a new notion of a "rationally connected singularity" might play a role.

A.8 Knecht, Amanda

I am interested in rationally connected varieties, especially non proper ones. Every smooth separably rationally connected (not necessarily proper) variety V contains a nonempty subset $V_0 \subset V$ characterized as the largest open subset such that if $v_1, \dots, v_m \in V_0$ are distinct closed points, then there is a very free curve in V_0 containing these as smooth points. No examples are know where $V_0 \neq V$. At this workshop, I would like to learn more techniques for constructing free curves with desired properties especially techniques involving deformation theory of curves and combs. These constructions would help prove that $V_0 = V$ in various cases.

A.9 Kollár, János

I am interested in most aspects of rationally connected varieties.

My recent work concerns rationally connected varieties which occur as quotients of Calabi-Yau varieties, see math.AG/0701466.

The paper settles the question of rational connectedness, but the more precise birational description is mostly unexplored.

I am also thinking about maps of curves to rationally connected varieties over finite fields, but I have no recent results. The main question I would like to see answered.

Assume that X is separably rationally connected over F_q, C is a smooth curve over F_q and Z a finite subscheme of S. Is it true that every morphism from Z to X extends to a morphism from C to X?

A.10 McKinnie, Kelly

My interest in the workshop "Rational curves on algebraic varieties" is to get to know the open questions in the area as well as the techniques used to study them. My background is in the Brauer group and I would be interested in problems and questions related to it, but I would also like to learn about new problems and areas of active research.

A.11 Miyaoka, Yoichi

(a) **On the initeness of the rational curves on a given surface of general type**: Let X be a minimal surface of general type and C a rational curve on X possibly with many singularities. I have obtained an explicit bound of CK_X under the condition that $c_2(X) < K_X^2$. The casse $c_2(X) \geq K_X^2$ is yet to be settled.

(b) **Fano manifolds with nef tangent bundles**: I have only a very partial result so far. However, the three-dimensional classification due to Peternell and others has been considerably simplified.

A.12 Starr, Jason

I have broad interests in many aspects of rational curves on algebraic varieties:

(1) the problem of proving there exist non-unirational rationally connected varieties,

(2) the weak approximation problem for rationally connected varieties over the function field of a curve,

(3) existence of sections of rationally simply connected fibrations over the function field of a surface,

(4) applications of sections to Galois cohomology of function fields,

(5) birational geometry of spaces of rational curves, encompassing the study of the ample and effective cones.

In addition to the excellent questions posted on the workshop website, I am also very interested in Ax's conjecure that PAC implies QAC in positive characteristic, the conjecture that non-uniruled projective manifolds have nonnegative Kodaira dimensions, and the problem of symplectic invariance of rational connectedness.

A.13 Treisman, Zachary

Rational connectedness has many useful descriptions. In particular is the following, which is due to Kollár.

(*) A smooth proper variety X is rationally connected if and only if for every finite collection of arcs (jets) $\{\gamma_i \in X_{m_i}\}$, there exists a dominant trivial family of rational curves $F : H \times {}^1 \to X$ such that for all $h \in H$ (1) the Taylor series of $F_h : {}^1 \to X$ at $\gamma_i(0)$ agrees with γ_i, and (2) $H^1({}^1, F_h^* T_X(-\sum m_i)) = 0$.

In this direction, inspired by the *variety of minimal rational tangents*, I am investigating families of rational curves that minimally surject onto the projective tangent bundle of a variety at a generic point, or $\widehat{X}_1$, and corresponding subvarieties in the higher projective arc spaces, $\widehat{X}_m$. These subvarieties of arcs, the so-called (by me) families of *minimally indicating rational arcs*, and their decomposition via singular loci might capture some information about the families of curves that generate them, and their explicit descriptions could make them amenable to concrete calculations.

That said, I'm more hoping to learn about the other tools that are used to study families of rational curves than I am to convince other people that studying arcs is a good idea.

A.14 Tschinkel, Yuri

I am interested in arithmetic questions over function fields of curves and surfaces. In particular, I hope to extend my joint work with B. Hassett on weak approximation for rationally connected varieties over function fields of curves to cover some new cases, specifically, to understand phenomena at singular fibers of rationally connected fibrations. Related questions include: finding free rational curves in prescribed homology classes, avoidence of special loci etc. I look forward to hearing the details of the recent work of de Jong and Starr proving weak approximation for hypersurfaces of small degree via the new notion of rational one-connectedness.

I am also interested in ample and effective cones of moduli spaces of curves.

A.15 Wittenberg, Olivier

My interest in this workshop is twofold. First, I hope to learn more about the purely geometric problems related to rationally connected (RC) varieties and rational curves on arbitrary algebraic varieties, the prospects for future work on these problems, and the techniques needed to attack them. Second, I am particularly interested in the many basic open questions which concern RC varieties over non-algebraically closed fields. Little is known about how rational points and rational curves on RC varieties behave in terms of the properties of the ground field k. Examples of such open questions include (with no claim to originality) the existence of rational points on RC varieties defined over C_1 fields, the existence of rational curves on real RC varieties which do not have real points, weak approximation for RC varieties defined over the function field of a complex curve, the existence of rational points (and weak approximation) on RC varieties defined over the function field of a real curve without real points, etc.

A.16 Zoghi, Masrour

Despite the title of the workshop, I will go ahead and commit the shameless crime of asking for genus of any sort. But, of course my request is completely meaningless because I know nothing and I'd absorb anything like a dry well; so, please just ignore this.