P-ADIC REPRESENTATIONS, MODULARITY, AND BEYOND

The American Institute of Mathematics

The following compilation of participant contributions is only intended as a lead-in to the ARCC workshop "p-adic representations, modularity, and beyond." This material is not for public distribution.

Corrections and new material are welcomed and can be sent to `workshops@aimath.org`

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A.1 Berger, Laurent

My interest in this workshop comes from the interplay between Breuil's p-adic Langlands program and the theory of p-adic representations. For example, recent advances in the p-adic Langlands program have allowed us to compute the reduction modulo p of some crystalline representations and in the other direction, the fact that the Colmez isomorphism should hold for those representations which become crystalline over an abelian extension of $\mathbf{Q}_p$ led us to work out the theory of Wach modules for those representations. It is my hope that new questions regarding the p-adic Langlands program will point the way toward new developments in the theory of p-adic representations.

A.2 Breuil, Christophe

Find the mod p local Langlands correspondence for GL2(F) where F is a finite extension of Qp different from Qp.

A.3 Buzzard, Kevin

Here is a list of some problems, some of which have essentially been answered already, and some of which may well be very hard. I (Buzzard) wrote this list and, by writing down as many questions as I could think of, I am of course at risk of exposing my ignorance of what is already known. However I will be happy to be told if I have asked a "question" that is already well-understood.

Global theory.

1) Let f be an overconvergent p-adic modular eigenform of weight κ. What is the "representation-theoretic" object that one should associate to f? One can pose this question in special or general circumstances. For example, what does one do in the following cases?

a) f classical of integer weight k, coming from a newform of level prime to p. Breuil may well have answered the question in this case: there is a natural lattice in the tensor product of $Symm^{k-2}$ and the smooth irreducible representation classically associated to f, and one could complete with respect to this lattice. Could one instead carry around the tensor product and the lattice? (this may well be known).

b) f classical coming from a newform of level $\Gamma_0(p)$ at p. Now one has to deal with the $\mathcal{L}$-invariant but again Breuil knows a lot.

c) f classical coming from a newform of level $\Gamma_1(p^n)$ and character of conductor p^n? Conductor p^r with $r < n$?

d) f overconvergent non-classical of integer weight k. Is the ordinary case easier? I believe Emerton knows something about this case. Is the infinite slope case different to the finite slope case?

e) f overconvergent of non-integer weight. Again is the ordinary case easier?

It would be nice to have a summary of what people have done here. Breuil has thought hard about the crystalline case but this basically forces the weight to be an integer. Can one expect analytic families of reprentation-theoretic data corresponding to analytic families of modular forms? If so then can one see $\mathcal{L}$-invariants here, as crystalline representations degenerate into semistable ones?

2) Chenevier has proved a p-adic Jacquet-Langlands theorem, relating finite slope overconvergent automorphic forms for $GL(2)$ and $D^{\times}$, the units of a definite quaternion algebra over $\mathbb{Q}$. His proof goes by "reducing to the classical case" (which is possible because overconvergent eigenforms of small slope are known to be classical in both settings) and the invoking the classical Jacquet-Langlands theorem (although arguably one could get away with "earlier versions" of the JL theorem). It's great that this strategy works, but it will never prove a p-adic functoriality statement in the cases where the classical one is not already known. Can one use p-adic methods to prove an instance of p-adic (or classical) functoriality that is not yet known? p-adic analysis is well-known to be easier than classical analysis, after all!

3) What is the correct generalisation of Coleman's θ^{k-1} map, taking overconvergent forms of weight $2-k$ to overconvergent forms of weight k? I (Buzzard) think I know what the answer to this should look like, and probably other people do too. A student of mine (Dan Snaith) found seven analogues of Coleman's map in the case of overconvergent cohomological forms on GSp_4 (or eight if you include the identity; note that the Weyl group for Sp_4 has size eight), I worked out some of the theory for $GL(3)$, and I think I know the general picture. My progress on this problem would be much faster if I found someone who was more familiar with the representation theory of reductive groups and especially the theory of Verma modules.

4) Kassaei has given a very simple proof of Coleman's result that overconvergent forms of small slope are classical. Which other Shimura varieties does this generalise to? I (Buzzard, joint with Toby Gee and Richard Taylor) have a good feeling about how far one can get in the Hilbert case. Specifically: one can almost certainly prove that overconvergent forms of small slope are classical if p is unramified in the base, but the general case throws up some subtle difficulties. What about the unitary case? Some cases may well be accessible and I believe Kassaei has done many cases where the variety is 1-dimensional. Note that Kassaei himself may be working on this already.

5) Katz knew what an overconvergent modular form of integer weight k was in the 1970s. One could even try and push through a geometric definition of an overconvergent modular form of weight κ for κ a p-adic integer (more precisely κ an element of Serre's weight space, the topological closure of the integers in weight space). But this geometric definition is not enough to construct an eigencurve: one needs a definition of an overconvergent modular form of weight κ for κ an arbitrary continuous map from $\mathbb{Z}_p^{\times}$ to $\mathbb{C}_p^{\times}$. Coleman gives an "ad hoc" definition which relies crucially on the Eisenstein series.

a) Is there a geometric definition?

b) How does one generalise Coleman's construction to the Hilbert modular case? There are no Eisenstein series here of non-parallel weight. On the other hand if one uses totally definite quaternion algebras one can certainly see p-adic familes of the correct dimension, so one surely should be able to see them on the Hilbert modular variety.

c) Other groups? I knew very little about unitary Shimura varieties but it would not surprise me if again one ran into similar problems. Note that Kassaei made some progress for certain Shimura curves in his thesis, but it is not clear (to Buzzard) how to generalise what he did to higher-dimensional cases.

6) Why has no-one thought about the global function field case? Even for $GL(2)$? I (Buzzard) checked that there is a notion of a not-too-supersingular Drinfeld module, that

such things have canonical subgroups and even that there was a rigid space parametrising not-too-supersingular Drinfeld modules. In particular many of the "key players" seemed to be there. I was too lazy to do any more though. I do not know anything about what to expect here—are there any interesting questions?

Local questions.

7) **Important.** Finite extensions of $\mathbb{Q}_p$ are being left behind and this concerns me greatly. Let K be a finite extension of $\mathbb{Q}_p$. Barthel and Livné wrote down a partial classification of the mod p representations of $GL(2, K)$. Breuil completed the classification in the case $K = \mathbb{Q}_p$, but not in the general case. Breuil's important result opened the door for results connecting mod p Galois representations and mod p representations of $GL(2)$ but *only for* $K = \mathbb{Q}_p$. The problem now is that more and more papers are appearing in this area, and all with $K = \mathbb{Q}_p$. Some of the methods in these papers work for any local field but the longer the key fundamental representation-theoretic questions left by Barthel and Livné remain unanswered, the worse the state of the literature will become, and we will have to read papers a decade down the line saying "x and y proved result z for $K = \mathbb{Q}_p$ but it is clear that the methods will work in the general case", which really translates down to "I think this should be OK and am not prepared to waste several months of my life rewriting the old papers in the generality that I need them". This is not good. I think that it is imperative that someone at least comes up with the correct conjecture about the general picture, so that people can say "Assuming conjecture X, we prove some great results about mod p local Langlands conjectures for a p-adic field. Note that currently conjecture X is currently only proved for $K = \mathbb{Q}_p$." This would be much better. We can prove conjecture X later. Is the general picture already conjectured? If so then the picture should be made as widely known as possible. There is a danger that things will get into an awful mess otherwise. Once this is done it should become very clear exactly when these papers really do need the assumption that $K = \mathbb{Q}_p$.

8) Colmez made some very daring statements during the 2004 Durham conference about a p-adic Local Langlands conjecture: for example he asked whether it was the case that isomorphism classes of n-dimensional p-adic representations of the absolute Galois group of $\mathbb{Q}_p$ (without any pst hypothesis) naturally bijected with isomorphism classes of unitary Banach space representations of $GL(n, \mathbb{Q}_p)$. This might be a little too much to ask. For example I think Breuil once told me that he was not sure how the "continuous Steinberg" representation of $GL(2)$ would fit into the picture. What is the general conjecture? This is well-known to be an important problem. Does one have to consider only irreducible representations, and add some kind of "supercuspidality condition" on the Banach space side (which may mean "not a subquotient of a p-adic principal series?"), and even then are we asking too much?

Again, it would perhaps be nice to have an overview of the known constructions (a Galois representation and the Banach space representation that we think it matches with), so at least we have something to stare at if we're hoping for inspiration.

9) What are the mod p representations of $GL(n, K)$ for K a p-adic field? Ron Livné told me that the principal series pose no surprises but that everything else was (at the time) thought to be much less tractable. Can we do any better nowadays?

10) Finally, a vague question about weight space, that might already be known. The recent breakthroughs in non-abelian Iwasawa theory, concluding with some explicit conjectures, make essential use of the fact that one can twist a 2-dimensional Galois representation by an irreducible representation of arbitrary (finite) dimension. This is what Kato et al do get "enough data points" to formulate their main conjecture. One interpretation of weight space is the 1-dimensional p-adic representations of the absolute Galois group of the rationals, which are unramified outside p. Is there a "higher weight space" which contains all the n-dimensional representations of this group? Is this even the right question? How does one formulate the objects that these Main Conjecture guys are looking at, in this setting? Is there some sort of link?

A.4 Calegari, Francesco

I'm interested in modularity of representations over imaginary quadratic fields. I'm also interested in a possible p-adic Jacquet–Langlands theorem. In the context of modular forms over imaginary quadratic fields this will be strictly stronger than any classical theorem as it must take into account torsion classes which don't correspond to classical automorphic forms.

A.5 Citro, Craig

I'm a graduate student, so I'm mostly looking to watch and learn, and have a chance to talk to experts and ask questions.

A.6 Crew, Richard

I'm interested in p-adic differential equations, p-adic cohomology theories, and the geometry of algebraic varieties in characteristic $p > 0$. My most recent work has been concerned with Berthelot's theory of arithmetic D-modules, and with various conjectures of Berthelot concerning the stability of holonomy under cohomological operations. I have also been thinking about the local p-adic Fourier transform, p-adic analogues of Laumon's results on the principle of stationary phase, and the local constants of the functional equation.

A.7 Dieulefait, Luis

Question 1: Modularity of rigid Calabi-Yau threefolds over Q:

a) As explained in a recent preprint of mine, by slightly improving a previous result of Dieulefait-Manoharmayum one concludes that any rigid Calabi-Yau threefold with good reduction at 5 is modular. Is the same statement true for the case of good reduction at 3? As explained in my preprint, the answer will be yes provided that the modularity lifting result of Diamond-Flach-Guo is extended to cover the case $p = k - 1$ (we need it for the case $k = 4$).

b) The following seems to be the easiest way of obtaining the proof of the modularity of every rigid Calabi-Yau threefolds over Q (among the tools one uses the full proof of the Taniyama-Shimura-Weil conjecture given by Breuil-Conrad-Diamond-Taylor, but one works at 5 where the ramification is smaller): start from the following known fact: the conductor at 5 of the family of Galois representations attached to a rigid Calabi-Yau threefold defined over Q is at most 25. This was first proved by Serre under the assumption of modularity, and I observed in a recent paper (published in Exp. Math.) that a proof without this assumption can be obtained, using among other things the potential modularity of the family of representations (proved by Taylor). Observe that the conductor of the family means the conductor of each ℓ-adic representation in the family except for $\ell = 5$. The fact that the value of the conductor is independent of ℓ is far from trivial: it was proved by R. Taylor (strong compatibility) as a consequence of his results on potential modularity. One can also use potential modularity to translate this bound for the conductor at 5 into a control of the local behavior at 5 of the 5-adic representation in the family. After doing this, the question is: is this local behavior good enough in order that a suitable available modularity lifting result applies? Namely, by the work of Breuil-Conrad-Diamond-Taylor we know that the residual mod 5 representation is either reducible or modular. So, if the 5-adic representation is good enough in order to apply some modularity lifting result, one can deduce from this the modularity of any rigid Calabi-Yau threefold over Q.

Question 2: One of the main tools used in the recent proof of some cases of Serres conjecture, both for the proof of existence of compatible families (1) (Dieulefait-Taylor) and for the bounding from above of minimal universal deformation rings in terms of f.f.c.i. modular rings after base change (2) (Dieulefait, Khare-Wintenberger), is the potential modularity result of R. Taylor. Observe that the proof of cases of Serres conjecture is based on the possibility of switching the residual characteristic to move to a place where residual modularity (or reducibility) is known (and iterations of this by induction).

Together with modularity liftings results la Wiles, this switching is just the combination of the above result (1) and the result of existence of minimal lifts which follows tautologically from (2) and a result of Boeckle (in fact Boeckle had proved, using his local-to-global principle, existence of minimal p-adic lifts precisely under the assumption that a bound from above such as (2) existed!).

So, since the potential modularity result of Taylor has played such an essential role in the construction of a minimal p-adic deformation, one could attack with similar techniques more general deformation problems (in particular, deformation problems with local conditions). Generalizations are possible in two directions:

a) First, always using potential modularity, more properties of modular forms can be translated to residual Galois representations. As long as the local conditions are good enough so that (by the local-to-global principle of Boeckle) Boeckles results can be applied (and give $\dim R > 0$), by the same reasoning as above all that is needed is an upper bound for the universal deformation ring, and this is obtained from potential modularity.

b) Second, one key point in Taylors potential modularity result is the use of a result of Moret-Bailly that allows him to contruct a geometric deformation over a number field satisfying some specified local conditions. If we forget about modularity for a while, in some cases this method is just giving us the potential resolution of the deformation problem (first solved by Ramakrishna) existence of crystalline p-adic deformations. Since the result of Moret-Bailly is very general, one could try to generalize these ideas and produce potential solutions to other deformation problems by this method.

So, what I propose is to explore these generalizations.

Final Remark: by combining it with a new result of mine relating existence of solutions with existence of potential solutions for a large class of deformation problems with local conditions (under some assumptions on the number field F) the strategy proposed in (b) above (construction of a potential solution to a deformation problem) can be very useful.

A.8 Emerton, Matthew

One of my main interests is to investigate further the local-global interaction in the p-adic Langlands program. This interaction is starting to become quite well understood in the case of the group $\mathrm{GL}_{2/\mathbb{Q}}$, and it is natural to wonder whether the phenomena observed in this case persist for more general groups/ground fields.

In order to facilitate such investigations, it may be helpful to recall something about the status of the theory for $\mathrm{GL}_{2/\mathbb{Q}}$ (and hence also the local theory for $\mathrm{GL}_{2/\mathbb{Q}_p}$, on which the global theory for $\mathrm{GL}_{2/\mathbb{Q}}$ heavily relies). In particular, I will recall some of the developments that were announced in the Montreal conference in September, and some of the developments that have taken place since then. While my description of recent (unpublished) work, and of work in progress, should be taken as provisional, I hope that it will be useful to the workshop participants.

Local theory

Let E be a finite extension of $\mathbb{Q}_p$ (this will be the field of coefficients). I think most experts now agree that there should be a local p-adic Langlands correspondence which would associate to any two dimensional continuous representation V of $G_{\mathbb{Q}_p}$ over E a corresponding unitary admissible Banach representation $B(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E. Although no definitive axiomatization of this conjectural correspondence has been written down in the literature, the experts have a reasonable feeling for at least some of the properties that it should satisfy.

Perhaps more importantly, it seems the correspondence is rather close to being constructed. This is ongoing work of Colmez, building on work that he presented at Montreal, and incorporating a deformation-theoretic strategy suggested by Kisin. I will now try to be a little more precise.

From $G_{\mathbb{Q}_p}$ to $\mathrm{GL}_2(\mathbb{Q}_p)$ Using the theory of (ϕ, Γ)-modules, Colmez has introduced a functor which starts with a $G_{\mathbb{Q}_p}$-representation over E, and produces a unitary Banach space representation over E of the Borel subgroup B of $\mathrm{GL}_2(\mathbb{Q}_p)$. The construction and basic properties of this functor are explained in two manuscripts of Colmez (the "représentations semi-stables" and "représentations trianguline" manuscripts), a manuscript of Berger and Breuil (the "représentations cristallines" manuscript), and a manuscript of Berger ("Représentations modulaires $\cdots$"). In the case when V is an irreducible two-dimensional trianguline representation, this functor can be combined with more explicit constructions from p-adic analysis to produce the desired unitary admissible $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach representation $B(V)$. (The role of the explicit p-adic analysis construction is to extend the action from the Borel B to all of $\mathrm{GL}_2(\mathbb{Q}_p)$.)

If V is two-dimensional and reducible, then in many cases Breuil has constructed the required representations $B(V)$. More recently (unpublished) I believe that I can construct them in all cases. It may be that Kohlhaase can also construct them, as a special case of a more general theory that he is developing – I am not sure. (When V is reducible, the representation $B(V)$ is supposed to be topologically reducible, and one knows what the corresponding Jordan-Hölder factors should be. The problem is to construct the appropriate extension.)

When V is potentially semi-stable but not trianguline, candidates for $B(V)$ have been given by Breuil and Strauch (as was explained by Strauch at Montreal). But as far as I know, it has not yet been proved that these $B(V)$ are irreducible admissible (or even non-zero).

From $\mathrm{GL}_2(\mathbb{Q}_p)$ to $G_{\mathbb{Q}}$ In his talk in Montreal, Colmez explained the construction of a functor that I will denote MF (Colmez's "Montreal functor") which passes from the $\mathrm{GL}_2(\mathbb{Q}_p)$ side to the $G_{\mathbb{Q}_p}$-side. In fact, it only relies on the B-structure (not the full $\mathrm{GL}_2(\mathbb{Q}_p)$-structure), and is essentially inverse to the functor discussed in the preceding subsection.

An important feature of both the functors of Colmez (the Montreal functor and its inverse) is that they are compatible with change of coefficients, and work with p-adic or mod p coefficients just as well as with p-adic coefficients.

Mod p local Langlands Let $\mathbb{F}$ be the residue field of $\mathcal{O}_E$ (the ring of integers in our coefficient field E). In his papers on mod p and p-adic reps. of $\mathrm{GL}_2(\mathbb{Q}_p)$, Breuil has in particular described a mod p local Langlands that to any continuous two-dimensional semi-simple representation $\overline{\rho}$ of $G_{\mathbb{Q}_p}$ over $\mathbb{F}$ associates a certain semi-simple smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\overline{\pi}(\overline{\rho})$ over $\mathbb{F}$.

If $\overline{\rho}$ is irreducible, then so is $\overline{\pi}(\overline{\rho})$. If $\overline{\rho}$ is a sum of two characters, then $\overline{\pi}(\overline{\rho})$ is (essentially) a direct sum of two principal series representations (ordinary representations in the sense of Barthel and Livné).

However, it has become clear over time to the experts that it is possible, and important, to enhance this correspondence to handle the case of $\overline{\rho}$ that are reducible but non-semisimple. More precisely, if $\overline{\rho}$ is reducible but not semi-simple, one would like to find a matching $\overline{\pi}$ that is reducible but not semi-simple, such that $\overline{\rho}^{\mathrm{ss}}$ and $\overline{\pi}^{\mathrm{ss}}$ match according to Breuil's semi-simple correspondence.

Following Breuil, one can try to stipulate this correspondence "by hand" – to do this one has to make a computation of the extension classes on the $\mathrm{GL}_2(\mathbb{Q}_p)$ side and check that they match with those on the Galois side. (This problem is similar to that of the construction of the Banach representations $B(V)$ for reducible p-adic representations V.) I believe that I can do this. (A caveat: in certain cases there can be more extensions on the $\mathrm{GL}_2(\mathbb{Q}_p)$-side than on the $G_{\mathbb{Q}_p}$-side, so that the association $\overline{\pi} \mapsto \overline{\rho}$ is many-to-one in these cases.)

Using Colmez's Montreal functor, one can raise the mod p correspondence to a more theoretical level, by requiring that in fact $\mathrm{MF}(\overline{\pi}) = \overline{\rho}$. Breuil's semi-simple correspondence does indeed satisfy this requirement (as Berger has shown), and it seems that the "non-semisimple" correspondence discussed in the preceding paragraph also satisfies this requirement.

Deformation theory As in the preceding section, let $\overline{\pi}$ and $\overline{\rho}$ be matching objects on the $\mathrm{GL}_2(\mathbb{Q}_p)$ and $G_{\mathbb{Q}_p}$-sides respectively; so $\mathrm{MF}(\overline{\pi}) = \overline{\rho}$. One can consider the deformation functors $\mathrm{Def}(\overline{\pi})$ and $\mathrm{Def}(\overline{\rho})$ on the category of local Artinian thickenings of $\mathbb{F}$, classifying isomorphisms classes of deformations of $\overline{\pi}$ and of $\overline{\rho}$ respectively. Since MF is compatible with change of coefficients, it induces a natural transformation $\mathrm{Def}(\overline{\pi}) \to \mathrm{Def}(\overline{\rho})$.

0.1. **Expectation.** *The preceding natural transformation induces a natural isomorphism of deformation functors*

$$\mathrm{Def}(\overline{\pi}) \xrightarrow{\sim} \mathrm{Def}(\overline{\rho}).$$

In fact one has to be a little careful in those cases where the correspondence $\overline{\pi} \mapsto \overline{\rho}$ is not one-one, and in some of these cases the preceding expectation should probably be modified. More precisely, I expect that one will have to make such a modification in the case when $\overline{\rho}$ is a (possibly split) extension of χ by ψ (χ and ψ being a pair of $\mathbb{F}$-valued characters of $G_{\mathbb{Q}_p}$) with $\chi\psi^{-1}$ equal to the mod p cyclotomic character – and only in this case.

Because I am glossing over some points (such as that discussed in the preceding paragraph), I have not wanted to label the expectation as an official conjecture. I should also point out that as far as I know it was first proposed by Kisin, at the Montreal conference. Finally, it seems that Colmez is very close to proving it, so that with any luck it will soon be a theorem.

Note that this expectation, if true, establishes the p-adic local Langlands correspondence for two-dimensional $G_{\mathbb{Q}_p}$-representations: if V is such a representation, then one can regard (some integral lattice in) V as a deformation of some $\overline{\rho}$. Using the isomorphism of deformation functors, one then finds a corresponding $\mathrm{GL}_2(\mathbb{Q}_p)$ representation deforming $\overline{\pi}$, which (after tensoring back up with $\mathbb{Q}_p$) gives the sought-after $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach $B(V)$.

As far as I know, Colmez's proof-in-progress follows a suggestion of Kisin: namely, one exploits the existence of the local correspondence in the trianguline case, and uses the fact that the trianguline points are Zariski dense in deformation space (via the "infinite fern" argument introduced by Gouvêa and Mazur).

One property that would not follow obviously from this deformation-theoretic construction of the local correspondence, but which is expected to be true, is the following.

0.2. **Expectation.** *If V is potentially semi-stable, then $B(V)$ has a non-zero subspace of locally algebraic vectors.*

This is known to be true if V is furthermore trianguline – it is a consequence of the explicit nature of the construction of $B(V)$ in that case. It seems that Colmez may actually be able to prove this for general potentially semi-stable V.

The image of the local correspondence Once the local correspondence is constructed, one can ask which admissible unitary Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ are actually of the form $B(V)$ for some two-dimensional $G_{\mathbb{Q}_p}$-representation V.

Since for reducible V the correspondence will be completely explicit, it suffices to consider the question for irreducible V. For such V the associated $\mathrm{GL}_2(\mathbb{Q}_p)$ representation $B(V)$ will be topologically irreducible, and furthermore will be "supersingular", in the sense that it is not a subquotient of the parabolic induction to $\mathrm{GL}_2(\mathbb{Q}_p)$ of a unitary character of B. (Note the these subquotients rather appear as Jordan-Hölder factors of the $B(V)$ for reducible V.) It is then natural to conjecture that every supersingular unitary Banach $\mathrm{GL}_2(\mathbb{Q}_p)$-representation B is of the form $B(V)$ for some V. Indeed, the representation V should be obtained as $\mathrm{MF}(B)$. The only thing that is missing in this part of the story is a proof that $\mathrm{MF}(B)$ is two-dimensional. (Equivalently, one doesn't yet know how to prove that the mod p reduction of B is necessarily one of the $\overline{\pi}$ appearing in the mod p local Langlands correspondence.)

Global theory

In this section I will briefly describe a global p-adic Langlands conjecture for $\mathrm{GL}_{2/\mathbb{Q}}$, and a result in the direction of this conjecture. Briefly put, the conjecture depends on the validity of Expectation 0.1 for its precise formulation. But provided that one does accept this expectation as valid, one can then go on and prove the conjecture in many cases.

Some notation Suppose now that $\overline{\rho}$ is a global residual Galois representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$, absolutely irreducible and modular, and let Σ denote a set of primes containing p, as well as all the ramified primes of $\overline{\rho}$. Let R denote the universal deformation ring of $\overline{\rho}$ parameterizing deformations that are unramified outside of Σ, and let R^{mod} denote the maximal "pro-modular" quotient of R. (Precisely: for any classical newform f over $\overline{\mathbb{Q}}_p$ whose level is divisible only by primes in Σ, with associated residual representation isomorphic to $\overline{\rho}$, the p-adic Galois representation attached to f gives rise to a homomorphism $R \to \overline{\mathbb{Q}}_p$. We let $\wp_f$ denote the kernel of this homomorphism, and define $R^{\mathrm{mod}} := R/\bigcap_f \wp_f$, the intersection ranging over all such newforms f.)

Write $H^1(\Sigma) := \varinjlim_N H^1_{\text{ét}}\big(X(N), \mathcal{O}_E\big)$, where $X(N)$ denotes the modular curve over $\mathbb{Q}$ of full level N, and N ranges over all levels divisible only by primes in Σ; note that $H^1(\Sigma)$ is naturally a smooth representation of $G := \prod_{\ell\in\Sigma} \mathrm{GL}_2(\mathbb{Q}_\ell)$, and is also equipped with actions of the Hecke algebra $\mathbb{T}$ generated by the Hecke operators T_ℓ for $\ell \notin \Sigma$, and of $G_{\mathbb{Q}}$, which commute with one another and with the action of G. If $\mathfrak{m}$ denotes the maximal ideal of $\mathbb{T}$ corresponding to $\overline{\rho}$, then there is an isomorphism $R^{\mathrm{mod}} \xrightarrow{\sim} \mathbb{T}_{\mathfrak{m}}$. Localizing $H^1(\Sigma)$ at $\mathfrak{m}$ we thus obtain an $R^{\mathrm{mod}}[G_{\mathbb{Q}} \times G]$-module which we denote by $H^1(\Sigma)_{\overline{\rho}}$. Finally, we let $\widehat{H}^1(\Sigma)$ denote the p-adic completion of $H^1(\Sigma)$; this is again an $R^{\mathrm{mod}}[G_{\mathbb{Q}} \times G]$-module (but note the the G-action is no longer smooth). As a final piece of notation, let I be a minimal prime in R^{mod}, and let $\widehat{H}^1(\Sigma)_{\overline{\rho}}[I]$ denote the I-torsion submodule of $\widehat{H}^1(\Sigma)_{\overline{\rho}}$; this is an $(R^{\mathrm{mod}}/I)[G_{\mathbb{Q}} \times G]$-module.

Global p-adic Langlands Fix a minimal prime I of R^{mod}, and let $V_{R^{\mathrm{mod}}/I}$ be the universal deformation of $\overline{\rho}$ over R^{mod}/I. Assuming that Expectation 0.1 is true, we obtain a corresponding $\mathrm{GL}_2(\mathbb{Q}_p)$-representation Π_p over R^{mod}/I. One may also apply the classical local Langlands correspondence to $V_{R^{\mathrm{mod}}/I}$ restricted to $G_{\mathbb{Q}_\ell}$ for each $\ell \in \Sigma \setminus \{p\}$, to obtain $\mathrm{GL}_2(\mathbb{Q}_\ell)$-representations Π_ℓ over R^{mod}/I. Write $(R^{\mathrm{mod}}/I)' := \mathrm{Hom}_{\mathrm{cont}}(R^{\mathrm{mod}}/I, \mathcal{O}_E)$ (continuous $\mathcal{O}_E$-linear maps).

0.2.1. **Conjecture.** *There is an R^{mod}/I-linear and $G_{\mathbb{Q}} \times G$-equivariant isomorphism*

$$\widehat{H}^1(\Sigma)_{\overline{\rho}}[I] \xrightarrow{\sim} V_{R^{\mathrm{mod}}/I} \otimes \big(\widehat{\bigotimes}_{\ell\in\Sigma} \Pi_\ell\big)\widehat{\otimes}(R^{\mathrm{mod}}/I)'.$$

(On the right hand side, $G_{\mathbb{Q}}$ acts on the first factor, G acts on the second factor, and the action on $(R^{\mathrm{mod}}/I)'$ is trivial. The various tensor products and completed tensor products are taken over R^{mod}/I.)

In the direction of this conjecture, one has the following result.

0.2.2. **Theorem.** *Assume the following two hypotheses:*

A. *The restriction $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ cannot be written as a (possibly trivial) extension of χ by ψ, with $\chi\psi^{-1}$ either trivial or equal to the mod p cyclotomic character.*
B. *Expectation 0.1 is valid for $\overline{\rho}_{|G_{\mathbb{Q}_p}}$.*

Then Conjecture 0.2.1 holds for $\overline{\rho}$.

Hypothesis (1) of the theorem is of a technical nature; it doesn't seem intrinsic to the method of proof. (In particular, once one makes the correct formulation of Expectation 0.1 in the case when $\chi\psi^{-1}$ equals the mod p cyclotomic character, I expect that the theorem will admit an extension to this case.) Hypothesis (2) is obviously essential – indeed the object Π_p appearing in the target of the isomorphism of Conjecture 0.2.1 depends on the isomorphism of Expectation 0.1 for its definition.

Applications Conjecture 0.2.1, and consequently also Theorem 0.2.2, has various corollaries and applications. For example we obtain the following result in the direction of the Fontaine-Mazur conjecture.

0.2.3. **Corollary.** *Suppose that $\overline{\rho}$ satisfies hypothesis (1) and (2) of Theorem 0.2.2. Let V be a lift of $\overline{\rho}$ to a two-dimensional continuous representation of $G_{\mathbb{Q}}$ over E, and suppose that V is potentially semi-stable with distinct Hodge-Tate weights, and pro-modular. Suppose furthermore that Expectation 0.2 holds for V (e.g. V is trianguline). Then V is a twist of a representation arising from a classical newform of weight $k \geq 2$.*

We also have the following result, which was (essentially) conjectured by Kisin.

0.2.4. **Corollary.** *Suppose that $\overline{\rho}$ satisfies hypothesis (1) and (2) of Theorem 0.2.2. Let V be a lift of $\overline{\rho}$ to a two-dimensional continuous representation of $G_{\mathbb{Q}}$ over E, and suppose that V is trianguline and pro-modular. Then V is a twist of a representation arising from an overconvergent p-adic eigenform of finite slope.*

By combining the original modularity lifting results of Taylor and Wiles with the "infinite fern" argument of Gouvêa and Mazur, Böckle has proved in many cases that if $\overline{\rho}$ is modular, then any lifting of $\overline{\rho}$ to characteristic zero is promodular. Using more recent strengthenings of the Taylor-Wiles result, together with the results of Khare and Wintenberger on Serre's conjecture, it seems likely that one can extend Böckle's result quite a bit further. Thus the assumption in the preceding corollaries that V be pro-modular should be a fairly mild one.

Conjecture 0.2.1 has other applications, for example to the theory of p-adic L-functions, and also to the structure of the mod p cohomology of modular curves. In particular, it predicts – in representation theoretic terms – the multiplicity with which $\overline{\rho}$ appears in the mod p cohomology of a modular curve of arbitrary level. It thus provides a representation-theoretic interpretation and strengthening of Serre's conjecture, and of classical mod p multiplicity one results.

Kisin's approach to Fontaine-Mazur In his lecture at Montreal, Kisin sketched an argument that combined (his modification of) the Taylor-Wiles method with local p-adic Langlands techniques to prove certain cases of the Fontaine-Mazur conjecture, as well as of the Breuil-Mezard conjecture. In particular, he outlined a proof of the following result.

0.2.5. **Theorem.** *Suppose that p is odd. Let $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be modular, and suppose furthermore that both $\overline{\rho}_{|G_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})}}$ and $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ are both irreducible. Then if V is any lift of $\overline{\rho}$ over E that is potentially crystalline and trianguline with distinct Hodge-Tate weights, then V is a twist of the representation arising from a classical newform of weight $k \geq 2$.*

The assumption on $\overline{\rho}_{|G_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})}}$ is made to ensure that the method of Taylor-Wiles applies. The assumptions on $\overline{\rho}_{|G_{\mathbb{Q}_p}}$ and V are made in order to be able to apply results about local p-adic Langlands program. If one admits Expectations 0.1 and 0.2 (or perhaps some slight variants), then it seems that Kisin can eliminate these latter assumptions in his theorem. (It may be worth emphasizing again that Expectations 0.1 and 0.2 seem close to being proved by Colmez.)

Thus the p-adic local Langlands program seems to provide (at least) two routes to the Fontaine-Mazur conjecture for two-dimensional Galois representations over $\mathbb{Q}$ with distinct Hodge-Tate weights: one passing through the Breuil-Mezard conjecture, and the other through global p-adic Langlands.

A.9 Gee, Toby

I am particularly interested in the relationships between modularity lifting theorems and the p-adic local Langlands correspondence. Specifically:

- is there an integral version of the p-adic local Langlands correspondence, compatible with a mod p correspondence for representations which are not necessarily semisimple?

- is it possible to prove cases of a p-adic correspondence for local fields other than $\mathbb{Q}_p$, and then to prove modularity lifting theorems for Hilbert modular forms (possibly of high non-parallel weight)?

- is it possible to prove a modularity lifting theorem in which the weight varies (other than in the ordinary case)?

A.10 Kassaei, Payman

I am interested in various advances that have been made on the Fontain-Mazur conjecture by Kisin and others. It would be useful to have a talk that surveys this progress and depicts a picture of what is done and what needs to be done. In particular, I am interested in learning the state of the art in Kisin's conjecture on the characterization of (twists of) Galois representations attached to finite slope overconvergent modular forms as Trianguline representations (and its relationship to p-adic Langlands). I am also inetersted to find out if there has been any progress on understanding various aspects of the theory for infinite slope overconvergent modular forms. My understanding is that Kisin and Emerton have some unpublished work on this (?). I would also like to hear about Buzzard-Gee-Taylor's recent work on the Artin conjecture in the (unramified?) Hilbert case.

A.11 Kedlaya, Kiran

As an organizer, I'm interested in all aspects of this workshop. In general, I'm hoping to see our time divided between pursuing goals in the short term (resolving particular problems, like the remaining cases of Serre's conjecture), the medium term (generalizing Serre's conjecture, investigating Breuil's conjectures concerning a p-adic Langlands correspondence for GL_2), and the long term (moving away from GL_2, at least on the p-adic side).

At present, my operational interests relevant to this workshop are in the area of p-adic Hodge theory (mostly from the point of view of (ϕ, Γ)-modules) and p-adic Langlands correspondences. Hence I'm particularly keen to find out what sorts of results in these areas are called for by applications to questions of modularity, then maybe prove some of them.

In addition, here are some questions that come to mind:

A. I'd like to understand Colmez's preprint on the unitary principal series well enough to say, for instance, whether it extends naturally to GL_n. In general, how much does Colmez's idea to classify representations based on the structure of their (ϕ, Γ)-modules over the Robba ring give you in the direction of a Langlands correspondence?
B. How does one properly do p-adic Hodge theory "in families"? The answer presumably tells one about the local geometry of "eigencurves". There are some ideas about this floating around (cf. work of Andreotti and Brinon, and Hartl in the equal-characteristic case), but I think Berger and Colmez have a good approach in mind.
C. Can one adapt the approach from Kisin's "Crystalline representations and F-crystals" to modularity problems in higher weight? (Note to self: ask Kisin what the status of this question is.)
D. Do the arguments for "weakly admissible implies admissible" given by Berger and Kisin fit into a common framework? More generally, is there a more natural way than Fontaine's to build the rings in p-adic Hodge theory and prove some of the basic results (e.g., in order to see how to work properly in families)? One approach: I'd like to see a development of p-adic Hodge theory based on de Rham-Witt complexes; the possibility of doing this is suggested by Hesselholt's work on the absolute de Rham-Witt complex of a scheme over $\mathbb{Z}_{(p)}$, whose cohomology inherits a Gauss-Manin connection that seems to represent "variation of p".
E. Buzzard's questions 1 and 5 (and for that matter most of his others too). In particular, I'm enamored of the idea of interpreting overconvergent modular forms as sections of some sort of "sheaf"; Iovita can probably give us an update on what he and Stevens have come with in this direction.

A.12 Kohlhaase, Jan

I am currently working on questions concerning the representation theoretic side of a p-adic local Langlands correspondence. In the attempt to give a definition of locally analytic Jacquet functors (different from that of M. Emerton) that can be used to examine extensions of locally analytic representations, there are some severe functional analytic problems to be solved. Namely, one needs to find sufficiently strong splitting theorems for p-adic nuclear Fréchet spaces. To avoid these obstacles one might try to find other candidates for a "good" derived category of locally analytic representations.

The unitary principal series representations of $GL_2(\mathbb{Q}_p)$ were treated by P. Colmez. Up to now, however, there does not seem to be any systematic approach to studying unitary representations in general. In the archimedean theory, of course, one has the machinery of Hilbert spaces and $\mathbb{C}^*$-algebras available. It might be worthwhile to think about alternatives in the nonarchimedean setting.

A.13 Ramakrishna, Ravi

My work over the last decade has been largely motivated by Serre's Conjecture. In particular I have been interested in finding evidence for Serre's conjecture by showing characteristic zero lifts of irreducible mod p Galois representations exist. I am naturally interested in the work being done related to modularity. Recently Spencer hamblen and I have proved that characteristic zero lifts exist for a large class of reducible indecomposable mod p Galois representations. From the work of Skinner-Wiles this gives modularity of these mod p representations, though at a larger level than the minimal one.

A.14 Ribet, Kenneth

I really qualify here as a non-expert. I hope to learn from this conference what we now can prove about Serre's conjecture and the Fontaine–Mazur conjecture. This means that I would hope to see laid out the general strategy as experts now see it along with the technical obstacles that prevent us from writing down a complete proof. As one of the most important techniques is "modularity lifting" à la Wiles, Taylor–Wiles, Skinner–Wiles, Kisin, etc., etc., I hope to learn how different approaches complement each other.

My best-known work is surely the level-lowering results that made possible the reduction of the "Strong conjecture" of Serre to the qualitative version that states simply that certain representations are modular. As far as I know, there is still a situation (for mod 2 representations, of course!) where one still does not know whether the level can be lowered as in the strong conjecture. Why is that? Have we not yet found the complete proof, or is the situation similar to that where certain Hecke algebras turned out not to be Gorenstein at the prime 2?

A.15 Savitt, David

In recent months, by work of Kisin, the study of modularity lifting theorems and the search for a p-adic Langlands correspondence have become definitively linked. Combined with the recent remarkable progress on Serre's conjecture by Khare and Wintenberger, we now know that any p-adic representation that looks like it should arise from a modular form of arbitrary weight ≥ 2 and level prime to $2p$ actually does, subject to the hypothesis that the residual representation is absolutely irreducible upon restriction to $G_{\mathbb{Q}(\sqrt{p*})}$ and to $G_{\mathbb{Q}_p}$. Each conjecturally unnecessary hypothesis in this statement raises a question:

A. What are the prospects for modifying the methods of Taylor-Wiles-Kisin to remove the hypothesis of absolute irreducibility on restriction to $G_{\mathbb{Q}(\sqrt{p*})}$ from modular lifting theorems?
B. Can Kisin's modularity lifting theorem in the potentially Barsotti-Tate case be extended to $p = 2$, to complete the proof of Serre's conjecture?
C. What refinements of Kisin's recent methods, and what extensions of the work of Berger-Breuil-Colmez for trianguline representations, will be necessary to weaken the hypotheses locally at p? Especially, what are the prospects for the latter?

The above is the situation, so to speak, for two-dimensional representations of $G_{\mathbb{Q}}$ and for $\mathrm{GL}_2(\mathbb{Q}_p)$. Replacing $\mathbb{Q}$ with a totally real field F, Kisin has a modularity theorem for two-dimensional potentially Barsotti-Tate representations of G_F, and Gee has proved many cases of Buzzard-Diamond-Jarvis's conjectural description of Serre weights (for F unramified at p). It is natural to ask how to approach weakening the hypotheses on these results.

I am not aware of any reason that the methods of Khare and Wintenberger should not, in principle, have some success in approaching Serre's conjecture for mod p representations of G_F. However, what input is necessary in order for this to go through? Here I am thinking mainly in terms of p-adic Hodge theory and modularity lifting theorems, but perhaps also results about abelian varieties over $\mathcal{O}_F$ with very little ramification, in order to get the induction started.

Then we have the fundamental question of p-adic Langlands correspondences for $\mathrm{GL}_2(F')$ where F' is a completion of F at a prime above p. It seems like there is a lot of ground to be made up here. See Buzzard's question #7. How would this relate to higher-weight modularity lifting theorems over F?

Finally, what are the prospects for advances on any aspects of these questions for GL_n?

A.16 Schein, Michael

As a finishing graduate student, I am particularly interested in hearing the open problems proposed by the other participants.

What is the relation between mod p and p-adic Hilbert modular forms defined using Shimura curves and the mod p and p-adic Hilbert modular forms of Andreatta-Goren, defined using Hilbert modular varieties? What is the correct analogue in the Shimura curve setting of the theta operators defined by Andreatta and Goren? How much of the recent progress on Serre's conjecture can be generalized to totally real fields?

A.17 Schneider, Peter

The search for a p-adic extension of the local Langlands program is a rather recent but fast developing area. The workshop will bring together more or less all people working in this and related directions. I hope that the opportunity for extensive discussions with these people will lead to new ideas. For me this includes in particular the still very mysterious connections between p-adic continuous representation theory of reductive groups and the theory of p-adic modular forms.

A.18 Schraen, Benjamin

I'm interested in the construction of a complete De Rham cohomology theory for towers of modular curves. As we have a complete singular cohomology

$$\varprojlim_n \varinjlim_r H^1(X(Np^r), \mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q}_p,$$

I hope there exists a good completion of the space

$$\varinjlim_r H^1_{DR}(X(Np^r))$$

which is an admissible $GL_2(\mathbb{Q}_l)$-representation. The hope is to have a comparison theorem between the two Banach spaces. It would be fine to study how the GL_2-representations of Breuil appear in this complete cohomology and it can be a nice tool to study local-global compatibilities. However I haven't significant results for the moment.

A.19 Taylor, Richard

I would particularly like to hear about (in some depth) the work of Kisin, of Khare-Wintenberger and of Breuil-Colmez.

A.20 Wintenberger, Jean-Pierre

See separate preprint.

A.21 Yazdani, Soroosh

My main interest in this workshop is the applications of Taylor-Wiles method to proving modularity of certain mod p representations of the Galois group of the rationals. Specifically, I am very interested in the recent advances made by Taylor, Khare, Winterburger, and Dieulefait, in proving parts of Serre's conjecture regarding the modularity of such representations. I am also interested in the extending some of these results to other number fields.