Glossary: Spectra of matrices: notation

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`http://aimath.org/mathresources/glossary/`

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$A(R)$
: $= A[\overline{R}]$.

$A[R]$
: The principal submatrix of A whose rows and columns are indexed by R (where $R \subseteq \{1, 2, \ldots, n\}$.)

$A(v)$
: $= A(\{v\})$.

$\alpha(G)$
: The vertex independence number of G, i.e., the largest number k for which a coclique with k vertices exists.

$c_\lambda(Q)$
: If T is a loop-tree and $Q \subseteq V(T)$, $c_\lambda(Q)$ is the number of components of $T(Q)$ that allow eigenvalue λ.

$C_\lambda(T)$
: $C_\lambda(T) = \max\{c_\lambda(Q) - |Q| \ : Q \subseteq V(T)\}$.

C_n
: Cycle on n vertices.

$\chi(G)$
: The chromatic number of G (provided G does not have loops), i.e., the smallest number of color classes of any vertex coloring of G.

$\Delta(G)$
: $\Delta(G) = \max\{p - q :$ there is a set of q vertices whose deletion leaves p paths.$\}$ (Note that an isolated vertex is a path of order 1 and G is simple.)

diam(G)
: The diameter of G, i.e., the maximum distance between any two vertices of G.

$E(G)$
: The energy of the simple graph G.

$\eta(G)$

$\eta(G) = \min\{\text{rank}(B) : B \in \mathcal{S}_\eta^F(G), F \text{ any field}\}$ (G simple.)

$\mathcal{G}(A)$

The simple graph of symmetric matrix A, i.e., the simple graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} | \ b_{ij} \neq 0 \text{ and } i \neq j\}$. Note that the diagonal of B is ignored in determining $\mathcal{G}(B)$.

$G - S$

The result of deleting all vertices of $S \subset V$ and their incident edges from G.

$G - v$

The result of deleting a vertex v of G and its incident edges.

$\overline{G}$

The complement of simple graph G, i.e., $\overline{G} = (V, \overline{E})$.

$G[S]$

The subgraph induced by $S \subset V$, the subgraph with vertex set S and edge set $\{\{i, j\} \in E \mid i, j \in S\}$.

$|G|$

The order of G. The number of vertices in G.

$\widehat{G}$

The simple graph obtained from G by removing all loops from G.

$G = (V, E)$

A graph, usually on $\{1, \ldots, n\}$.

V is the set of vertices, E is the set of edges. G is allowed to have loops and/or multiple edges.

$gM^F(G)$

$gM^F(G) = \max\{\text{gmult}_B(\lambda) : B \in \mathcal{S}^F(G), \lambda \in \sigma(B))\}$.

$\text{gmult}_A(\lambda)$

The geometric multiplicity of λ as an eigenvalue of A (i.e., the dimension of $\ker(A - \lambda I)$).

$H(G)$

The set of high degree vertices of simple graph G, i.e., the set of vertices of degree at least 3.

$\mathcal{H}(G)$

$\mathcal{H}(G) = \{B : B \text{ is a Hermitian } n \times n \text{ matrix over } \mathbb{C} \text{ and } \mathcal{G}(B) = G\}$. ($G$ is simple.)

$\mathcal{H}^+(G)$

$\mathcal{H}^+(G) = \{B \in \mathcal{H}(G) : B \text{ is positive semidefinite}\}$. ($G$ is simple.)

hmr(G)
: $\text{hmr}(G) = \min\{\text{rank}(B) : B \in \mathcal{H}(G)\}$. ($G$ is simple.)

hmr$^+$(G)
: $\text{hmr}^+(G) = \min\{\text{rank}(B) : B \in \mathcal{H}^+(G)\}$.

K_n
: The complete (simple) graph on n vertices.

$K_{p,q}$
: The complete (simple) bipartite graph on p and q vertices.

$M(G)$
: The maximum multiplicity of simple G, i.e., $M(G) = \max\{\text{mult}_A(\lambda) : \ A \in \mathcal{S}(G), \lambda \in \mathbb{R}\}$.

$M^F(G)$
: $M^F(G) = \max\{\text{mult}_B(\lambda) : B \in \mathcal{S}^F(G), \lambda \in \sigma(B))\}$. ($G$ simple.)

$M^+_\lambda(G)$
: $M^+_\lambda(G) = \max\{\text{mult}_B(\lambda) : B \in \mathcal{S}^+(G), \lambda \in \sigma(B))\}$. ($G$ simple.)

$M^\ell_\lambda(G)$
: $M^\ell_\lambda(G) = \max\{\text{mult}_B(\lambda) : B \in \mathcal{S}^\ell(G), \lambda \in \sigma(B))\}$. (It is necessary to distinguish between zero from nonzero eigenvalues because translation is no longer possible.)

mr(G)
: The minimum rank of simple graph G, i.e., $\text{mr}(G) = \min\{\text{rank}(A) : \ A \in \mathcal{S}(G)\}$.

mr$^\ell$(G)
: $\text{mr}^\ell(G) = \min\{\text{rank}(B) : B \in \mathcal{S}^\ell(G)\}$.

mr^F(G)
: $\text{mr}^F(G) = \min\{\text{rank}(B) : B \in \mathcal{S}^F(G)\}$. ($G$ is simple.)

mr$^+$(G)
: $\text{mr}^+(G) = \min\{\text{rank}(B) : B \in \mathcal{S}^+(G)\}$. ($G$ is simple.)

$\mu(G)$
: The **Colin de Verdière number** $\mu(G)$ of simple graph G is the maximum multiplicity of 0 as an eigenvalue among matrices L that satisfy:

  - L is a generalized Laplacian matrix of G.
  - L has exactly one negative eigenvalue (of multiplicity 1).
  - L satisfies the Strong Arnold Hypothesis.

mult$_A$(λ)
: The multiplicity of λ as a root of the characteristic polynomial of A (i.e., the algebraic multiplicity of λ if λ is an eigenvalue of A and 0 otherwise).

mvr(G)

The minimum vector rank of simple graph G, i.e., the minimum rank over all vector representations of G.

$\nu(G)$

The parameter $\nu(G)$ of simple graph G is defined to be the maximum multiplicity of 0 as an eigenvalue among matrices A that satisfy:

- $A \in \mathcal{S}(G)$.
- A is positive semidefinite.
- A satisfies the Strong Arnold Hypothesis.

$\nu^{\mathbb{C}}(G)$

The parameter $\nu^{\mathbb{C}}(G)$ is defined to be the maximum multiplicity of 0 as an eigenvalue among matrices A that satisfy:

- $A \in \mathcal{H}^{+}(G)$.
- A satisfies the complex Strong Arnold Hypothesis.

$\nu^{\mathbb{R}}(G)$

Same as $\nu(G)$

$P(G)$

The **path cover number** of simple graph G, i.e., the minimum number of vertex disjoint paths occurring as induced subgraphs of G that cover all the vertices of G.

P_n

The path on n vertices.

$\overline{R}$

The complement of R (universe may be set of edges in a graph or index set $\{1, \ldots, n\}$).

$r_v(G)$

$r_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G - v)$, the rank-spread of simple graph G at vertex v.

$\mathcal{S}(G)$

The set of symmetric matrices of simple graph G, i.e.,
$\mathcal{S}(G) = \{A \in S_n : \mathcal{G}(A) = G\}$.

$\mathcal{S}^{\ell}(G)$

$\mathcal{S}^{\ell}(G) = \{B \in S_n : b_{ij} \neq 0$ if and only if $ij \in E(G).\}$ Note G may have loops and the diagonal zero-nonzero pattern is restricted by G.

$\mathcal{S}^{F}(G)$

$\mathcal{S}^{F}(G) = \{B : B$ is a symmetric $n \times n$ matrix over F and $\mathcal{G}(B) = G\}$.

$\mathcal{S}^{+}(G)$

$\mathcal{S}^{+}(G) = \{B \in \mathcal{S}(G) : B$ is positive semidefinite$\}$. (G is simple.)

$\mathcal{S}_\eta^F(G)$

$\mathcal{S}_\eta^F(G) = \{B \in F^{n\times n} : B$ is a symmetric matrix over F, if $ij \notin E(G)$, then $b_{ij} = 0$ and $\forall i, b_{ii} \neq 0\}$. ($G$ is simple and no requirement that the entry corresponding to an edge be nonzero.)

S_n

The set of real symmetric $n \times n$ matrices.

$\sigma(A)$

The spectrum of A, i.e., the multiset of the n roots of the characteristic polynomial in the algebraic closure of F, where $A \in F^{n\times n}$.

$\theta(G)$

Clique covering number of G. The smallest number of cliques in a clique covering of G.

X_T

The matrix of indeterminates for a loop-tree T, i.e., for $i \leq j, i, j \in V(T), ij \in E(T)$, let x_{ij} be independent indeterminates and $(X_T)_{ij} = x_{ij}$ and $(X_T)_{ji} = x_{ij}$, and let the entries that do not correspond to edges be 0.

$\xi(G)$

The parameter $\xi(G)$ of simple graph G is the maximum multiplicity of 0 as an eigenvalue among matrices A that satisfy:

- $A \in \mathcal{S}(G)$.
- A satisfies the Strong Arnold Hypothesis.