# Glossary: Spectra of matrices: terminology

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`http://aimath.org/mathresources/glossary/`

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**2-connected**
: A graph of order at least 3 that does not have a cut-vertex.

**$(3, \ell)$-whirl ($\ell \geq 1$)**
: The tree on $n = 6\ell + 4$ vertices with 6 pendant paths $\alpha_i, \beta_i, \gamma_i$, each with $\ell$ vertices, for $i = 1, 2$.

**acyclic**
: A graph that does not contain any cycles. Also a matrix $A$ whose graph $\mathcal{G}(A)$ is acyclic.

**alexd511**
: Very nice site!

**alexd860**
: *definition needed*

**allows singularity**
: Graph $G$ allows singularity if there exists $B \in \mathcal{S}^{\ell}(G)$ that is singular.

**appropriate**
: A vertex $v$ of $T$ is appropriate if $T - v$ has at least two components that are pendent paths.

**center**
: The unique high degree vertex in a generalized star or pendent generalized star (if one exists).

**chordal**
: A graph that does not contain an induced cycle on four or more vertices.

**chromatic number**
The smallest number of color classes of any vertex coloring of $G$ (provided $G$ does not have loops), denoted $\chi(G)$.

**clique**
An induced subgraph $G'$ of a graph $G$ that has an edge between every pair of vertices of $G'$ (i.e., $G'$ is isomorphic to $K_n$ where $n = |G'|$).

**clique covering**
A set of subgraphs of $G$, each of which is a clique and such that every edge of $G$ is contained in at least one of these cliques.

**clique covering number**
The smallest number of cliques in a clique covering of $G$, denoted $\theta(G)$.

**coclique**
A set of vertices $S$ of graph $G$ such that the induced subgraph $G[S]$ has no edges.

**Colin de Verdière parameters** $\mu, \nu = \nu^{\mathbb{R}}, \nu^{\mathbb{C}}$
$\mu(G)$ is the maximum multiplicity of 0 as an eigenvalue among matrices $L$ that satisfy:

- $L$ is a generalized Laplacian matrix of $G$,
- $L$ has exactly one negative eigenvalue (of multiplicity 1),
- $L$ satisfies the Strong Arnold Hypothesis.

$\nu(G)$ is defined to be the maximum multiplicity of 0 as an eigenvalue among matrices $A$ that satisfy:

- $A \in \mathcal{S}(G)$,
- $A$ is positive semidefinite,
- $A$ satisfies the Strong Arnold Hypothesis.

$\nu^{\mathbb{C}}(G)$ is defined to be the maximum multiplicity of 0 as an eigenvalue among matrices $A$ that satisfy:

- $A \in \mathcal{H}(G)$,
- $A$ is positive semidefinite,
- $A$ satisfies the complex Strong Arnold Hypothesis.

**Colin de Verdière type-parameter** $\xi$
$\xi(G)$ is the maximum multiplicity of 0 as an eigenvalue among matrices $A$ that satisfy:

- $A \in \mathcal{S}(G)$
- $A$ satisfies the Strong Arnold Hypothesis.

**color class**
One of the cocliques in a vertex coloring.

**complete subgraph**
See clique.

**complex Strong Arnold Hypothesis**

A Hermitian matrix $M$ satifies the complex Strong Arnold Hypothesis provided there does not exist a nonzero Hermitian matrix $X$ satisfying:

- $MX = 0$.
- $M \circ X = 0$.
- $I \circ X = 0$.

**component**

A maximum connected (induced) subgraph. Same as connected component.

**connected**

There is a path from any vertex to any other vertex. (A graph of order one is connected whether or not it has a loop.)

**connected component**

A maximum connected (induced) subgraph.

**contraction**

A graph obtained from $G$ by identifying two adjacent vertices of $G$, suppressing any loops or multiple edges that arise in this process.

**cut-vertex**

A vertex $v$ of a connected simple graph $G$ such that $G - v$ is disconnected. More generally, $v$ is a cut-vertex of a graph $G$ if $v$ is a cut-vertex of a component of $G$.

**decomposable**

A graph that can be expressed as a sequence of unions and joins of isolated vertices.

**degree sequence**

Let $G$ be a graph having vertices $v_1, \ldots, v_n$ of degrees $d_1 \leq d_2 \leq \cdots \leq d_n$. The degree sequence of $G$ is $(d_1, d_2, \ldots, d_n)$.

**delete (vertex or set of vertices)**

The result of deleting vertex $v$ of $G$ and its incident edges, denoted $G - v$. If $S \subset V$, $G - S$ is the result of deleting all vertices of $S$ and their incident edges from $G$.

**diameter**

$\operatorname{diam}(G)$ is the maximum distance between any two vertices of G.

**disconnected**

Not connected.

**distance (between two vertices)**

The number of edges in a shortest path between the vertices.

**double generalized star**

A tree that can be constructed from two generalized stars by joining their centers by an edge.

**double path**

A tree that can be constructed from two paths $T_1$ and $T_2$ each of order $\geq 3$ by joining a vertex of degree two in $T_1$ to a vertex of degree two in $T_2$ by an edge.

**dual**

The dual of a plane embedding of a planar graph $G$ is obtained as follows: Place a new vertex in each face of the embedding; these are the vertices of the dual. Two dual vertices are adjacent if and only if the two faces of $G$ share an edge of $G$.

Note that the dual of a planar graph is not well-defined in general: the graph can have two embeddings with different duals.

**energy of a graph**

The sum of the absolute values of the eigenvalues of the adjacency matrix of the graph.

**generalized Laplacian matrix**

(of simple $G$) A symmetric matrix $L$ such that $\mathcal{G}(L) = G$ and all off-diagonal entries of $L$ are non-positive.

**generalized star**

A tree that has at most one high degree vertex.

**geometric multiplicity**

(of $\lambda$ as an eigenvalue of $B$) The dimension of $\ker(B - \lambda I)$, denoted $\mathrm{gmult}_B(\lambda)$.

**graph**

A set of vertices and a set of edges joining vertices. A graph may allow multiple edges and/or loops. A simple graph allows neither. Most graphs under discussion are simple.

**graph (of $B \in S_n$)**

The simple graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} |\ b_{ij} \neq 0 \text{ and } i \neq j\}$, denoted $\mathcal{G}(B)$. Note that the diagonal of $B$ is ignored in determining $\mathcal{G}(B)$.

**$H$-free**

A subgraph $G_1$ of $G$ is $H$-free if $G_1$ has no vertex in $H$ ($H \subset V(G)$).

**Hermitian minimum rank**

$\mathrm{hmr}(G) = \min\{\mathrm{rank}(B) : B \in \mathcal{H}(G)\}$. ($G$ is simple.)

**Hermitian positive semidefinite minimum rank**

$\mathrm{hmr}^+(G) = \min\{\mathrm{rank}(B) : B \in \mathcal{H}^+(G)\}$. ($G$ is simple.)

**high degree vertex**

A vertex of degree at least 3. The set of high degree vertices of $G$ is denoted $H(G)$.

**hyperenergetic graph**

A simple graph with $n$ vertices whose energy is greater than the energy of the complete graph $K_n$.

**independent set of vertices**

A set of vertices $S$ of graph $G$ such that the induced subgraph $G[S]$ has no edges.

**induced subgraph**

The subgraph $G[S]$ of $G = (V, E)$ induced by $S \subset V$ is the subgraph with vertex set $S$ and edge set $\{\{i, j\} \in E \mid i, j \in S\}$.

**Inverse Eigenvalue Problem of a graph (IEP-G)**

To characterize the possible spectra of matrices in $\mathcal{S}(G)$. ($G$ is simple.)

**linear 2-tree**

A 2-connected graph $G$ that can be embedded in the plane such that the graph obtained from the dual of $G$ after deleting the vertex corresponding to the infinite face is a path.

**loop-tree**

The graph $T$ is a loop-tree if $\widehat{T}$ is a (simple) tree.

**low degree vertex**

A vertex of degree less than 3.

**matrix**

The rank-spread of simple graph $G$ at vertex $v$ is $r_v(G) = \text{mr}(G) - \text{mr}(G - v)$.

**matrix of indeterminates**

For a loop-tree $T$, define it its matrix of indeterminates $X_T$ as follows:
For $i \leq j, i, j \in V(T), ij \in E(T)$, let $x_{ij}$ be independent indeterminates and $(X_T)_{ij} = x_{ij}$ and $(X_T)_{ji} = x_{ij}$, and let the entries that do not correspond to edges be 0.

**maximum multiplicity**

$M(G) = \max\{\text{mult}_B(\lambda) : B \in \mathcal{S}(G), \lambda \in \mathbb{R}\}$ (over $\mathbb{R}$). Over field $F$, $M^F(G) = \max\{\text{mult}_B(\lambda) : B \in \mathcal{S}^F(G), \lambda \in \sigma(B))\}$. $gM^F(G) = \max\{\text{gmult}_B(\lambda) : B \in \mathcal{S}^F(G), \lambda \in \sigma(B)).\}$ ($G$ is a simple graph.)

**minimum number of distinct eigenvalues**

(of simple graph $G$) $q(G) = \min\{q(B) : B \in \mathcal{S}(G)\}$ where $q(B)$ denotes the number of distinct eigenvalues of $B$.

**minimum rank**

$\text{mr}(G) = \min\{\text{rank}(B) : B \in \mathcal{S}(G)\}$ (over $\mathbb{R}$). Over field $F$, $\text{mr}^F(G) = \min\{\text{rank}(B) : B \in \mathcal{S}^F(G)\}$. ($G$ is a simple graph.)

**minimum rank problem**

Determine the minimum rank among real symmetric matrices whose zero-nonzero pattern of off-diagonal entries is described by a given simple graph $G$.

**minimum vector rank**

For any graph $G$, the minimum rank over all vector representations of $G$, denoted by $\text{mvr}(G)$.

**minor**

A graph obtained from $G$ by performing a series of deletions of edges, deletions of isolated vertices, and/or contraction of edges.

**minor monotone**

A graph parameter $\zeta$ such that for any minor $G'$ of $G$, $\zeta(G') \leq \zeta(G)$.

**monotone on induced subgraphs**

A graph parameter $\zeta$ such that for any induced subgraph $H$ of $G$, $\zeta(H) \leq \zeta(G)$.

**multiplicity**

(of eigenvalue $\lambda$ of square matrix $B$) The multiplicity of $\lambda$ as a root of the characteristic polynomial of $B$ (i.e., the algebraic multiplicity of $\lambda$), denoted $\text{mult}_B(\lambda)$.

**order**

The order of $G$, denoted $|G|$, is the number of vertices of $G$. All graphs are finite (finite number of vertices and finite number of edges).

**ordered multiplicity list**

$(m_1, \ldots, m_q)$ where the distinct eigenvalues of $B \in S_n$ are $\breve{\beta}_1 < \cdots < \breve{\beta}_q$ with multiplicities $m_1, \ldots, m_q$.

**Parter-Wiener (PW) vertex**

Vertex $k$ is a Parter-Wiener (PW) vertex of $B$ for eigenvalue $\lambda$ if
$\text{mult}_{B(k)}(\lambda) = \text{mult}_B(\lambda) + 1$.

**path cover number**

The minimum number of vertex disjoint paths occurring as induced subgraphs of simple graph $G$ that cover all the vertices of $G$, of $G$, denoted $P(G)$.

**pendent generalized star**

A connected induced subgraph $S$ of $G$ such that:

- There is exactly one vertex $v$ of $S$ that is a high degree vertex of $G$,
- $G - v$ has $k + 1$ components and exactly $k$ of the components of $G - v$ are pendent paths of $v$,
- $S$ is induced by the vertices of the $k$ pendent paths and $v$.

**pendent path**

A path $P$ in $G$ is a pendent path of vertex $v$ if $P$ is a component of $G-v$ and (in $G$) $P$ is connected to $v$ by one of its end-points.

**Pharmb901**

Very nice site!

**Pharmd220**

Very nice site!

**Pharmd267**

Very nice site!

**Pharmf531**

Very nice site!

**positive semidefinite**

A real symmetric matrix $B$ such that for all $\mathbf{x} \in \mathbb{R}^n, \mathbf{x}^T B\mathbf{x} \geq 0$. More generally, a matrix $B \in \mathbb{C}^{n\times n}$ such that for all $\mathbf{x} \in \mathbb{C}^n, \mathbf{x}^T B\mathbf{x} \geq 0$ (a complex positive semidefinite matrix is necessarily Hermitian).

**positive semidefinite minimum rank**

$\mathrm{mr}^+(G) = \min\{\mathrm{rank}(B) : B \in \mathcal{S}^+(G)\}$. ($G$ is simple.)

**principal submatrix**

If $R \subseteq \{1, 2, \ldots, n\}$ and $B \in F^{n\times n}$, then the principal submatrix of $B$ defined by $R$ is the submatrix of $B$ whose rows and columns are indexed by $R$, denoted $B[R]$. $B(R)$ denotes the complementary principal submatrix obtained from $B$ by deleting the rows and columns indexed by $R$.

**rank**

(of vector representation $\vec{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\} \subset \mathbb{C}^m$) $\dim \mathrm{Span}(W)$.

**rank-strong**

A vertex $v$ of $G$ is rank-strong if $\mathrm{mr}(G) = \mathrm{mr}(G-v) + 2$.

**SAP**

Spectrally Arbitrary Pattern

**set of Hermitian matrices (of simple graph $G$)**

$\mathcal{H}(G) = \{B : B$ is a Hermitian $n \times n$ matrix over $\mathbb{C}$ and $\mathcal{G}(B) = G\}$.

**set of symmetric matrices (of simple graph $G$)**

$\mathcal{S}(G) = \{B \in S_n : \mathcal{G}(B) = G\}$ (over $\mathbb{R}$).

Over a field $F$, $\mathcal{S}^F(G) = \{B : B$ is a symmetric $n\times n$ matrix over $F$ and $\mathcal{G}(B) = G\}$.

**simple graph**

A graph that does not have multiple edges or loops.

**spectrum**

For a matrix $A \in F^{n\times n}$, the multiset of the $n$ roots of the characteristic polynomial in the algebraic closure of $F$, denoted $\sigma(A)$.

**Strong Arnold Hypothesis**

A symmetric real matrix $M$ is said to satisfy the Strong Arnold Hypothesis provided there does not exist a nonzero symmetric matrix $X$ satisfying:

- $MX = \mathbf{0}$.
- $M \circ X = \mathbf{0}$.
- $I \circ X = \mathbf{0}$.

**strong PW vertex**

Vertex $k$ is a strong PW vertex of $B$ for $\lambda$ if $k$ is a PW vertex of $B$ for $\lambda$ and $\lambda$ is an eigenvalue of at least three components of $\mathcal{G}(B) - k$.

**tree**

A tree is a connected acyclic simple graph.

**typical**

A vertex $v$ of $G$ is typical if $v$ has at least two low-degree neighbors.

**vector representation**

(of graph $G$) A set $\vec{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\} \subset \mathbb{C}^m$, where none of the $\mathbf{w}_i$ is the zero vector and for $i \neq j$, $\langle \mathbf{w}_i, \mathbf{w}_j \rangle \neq 0$ if vertices $i$ and $j$ are connected and $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = 0$ if $i$ and $j$ are not connected.

**vertex coloring**

A partition of the vertex set into cocliques.

**vertex independence number**

The largest number $k$ for which a coclique of $G$ with $k$ vertices exists, denoted by $\alpha(G)$.