Report on the Holomorphic Mappings Conference at The American Institute of Mathematics (July 30,2000-August 4, 2000) and the State of the Art in the Theory of Several Complex Variables

The theory of several complex variables is now 100 years old. It is a lively and rich area of mathematics, and it continues to grow. The wellspring of the subject resides in two fundamental results proved in 1906:

[Poincaré] The unit ball in $\mathbb{C}^2$ and the bidisc in $\mathbb{C}^2$ are not biholomorphically equivalent.

[Hartogs] There exists a bounded domain $\Omega \subseteq \mathbb{C}^n$ with the property that every holomorphic function on Ω analytically continues to a strictly larger domain $\widehat{\Omega}$.

The first of these results suggests that there is no Riemann mapping theorem in several complex variables. The second shows that not every domain in several variables is the natural domain of existence of a holomorphic function.

Today the subject area has exploded in diverse and unpredictable directions. Besides using ideas from function theory, several complex variables routinely employs techniques from differential geometry, partial differential equations, harmonic analysis, real analysis, operator theory, geometric measure theory, commutative algebra, and Lie group theory.

The purpose of this holomorphic mappings conference was to bring together about 40 recognized experts to share recent progress in subject areas that find their roots in Poincaré's theorem described above. Recent progress includes:

(1) Deep studies of mappings of real submanifolds in complex space and the establishing of an analytic relationship between formal holomorphic mappings and classical holomorphic mappings.

(2) A complete understanding of the theory of subelliptic estimates, and of the role of points of finite type in that theory, for the $\overline{\partial}$-Neumann problem.

(3) The use of techniques of symplectic geometry and the $\overline{\partial}$ equation to study Lagrangian submanifolds, rationally convex surfaces, and contact structures.

(4) Development of the importance of the Bergman kernel and Bergman metric in studying questions of function theory, holomorphic mappings, and embedding theorems.

(5) Substantial progress on the question of imbedding of CR manifolds.

(6) Investigations of complex dynamics in higher dimensions. Study of problems in quantum chaos.

(7) Development of a theory of automorphism groups of bounded domains, both in several and in infinitely many complex variables.

(8) Studies of pluripotential theory and the complex Monge-Ampère equation. Use of currents and other techniques of geometric measure theory.

(9) The use of the Bergman Carathéodory, Kobayashi, and other invariant metrics in studying questions of mappings and other aspects of geometric analysis.

(10) Applications of complex partial differential equation techniques to algebraic geometry. Use of the Ohsawa-Takegoshi extension theorem to study plurigenera.

(11) Clarification and completion of sharp results on the embedding of Stein manifolds.

The talks and discussions at our conference covered all these subjects and more. Participants and speakers included

Abate	Baouendi	Barrett	Bedford	Bonifant
Burns	Christ	Coman	D'Angelo	Diller
Duval	Ebenfelt	Fornaess	Fridman	Fu
Gaussier	Gavosto	Globevnik	Graham	Han
Haslinger	Isaev	Kim	Krantz	Li
Low	Ma	McNeal	Merker	Ohsawa
Peloso	Pinchuk	Polyakov	Prezelj	Rothschild
Shafikov	Sibony	Stout	Tumanov	Varolin
Weickert				

In fact our short list of topics represents just some of the most recent progress in a period of three decades that has seen an explosion of developments in the subject of several complex variables. First some history: in order to understand the phenomenon of Hartogs mentioned above, a program (the Levi problem) was initiated to determine an extrinsic geometric characterization of domains that are the natural domain of definition of some holomorphic function. The machinery of sheaf theory was developed to attack this problem. In the end, thanks to efforts of Oka, Bremerman, Cartan, Narasimhan, and others, some versions of the Levi problem were completely solved.

The tools developed in the past few decades have taken us far beyond what was possible with sheaf theory. Estimates for the $\overline{\partial}$-Neumann problem and the complex Monge-Ampère equation, subtle analysis of the Bergman kernel, the creation of integral formulas, new uses of geometric techniques, the use of currents, analysis of mappings, the application of invariant metrics, methods of dynamics, techniques of algebraic geometry, interactions with non-commutative geometry, and many other new ideas have elevated the subject to unprecedented levels of sophistication.

As we enter the new millenium, we see an exciting dynamic developing in our subject. The center of activity in this country is no longer at Princeton University. It is now focused in the Midwest, with strong workers at the University of Michigan, the University of Illinois, Purdue University, Indiana University, and many other institutions. There are also strong groups on the West coast, particularly in Berkeley, San Diego, and Seattle. One effect of this geographic movement is that the subject now interfaces with many new fields: symplectic geometry, Riemannian geometry, hard analysis, dynamical systems, differential equations, and classical complex function theory. Almost all of the important developments in linear partial differential equations since World War II—including local solvability and non-solvability, subellipticity, hypoellipticity, propagation of singularities, and pseudodifferential operators—grew out of studies in several complex variables. Today we see fundamental questions in dynamical systems, differential geometry, harmonic analysis, and many other fields that have developed from a symbiotic relationship with multivariable complex analysis.

Thirty-five years ago there were no texts in several complex variables; now there are many. There were no journals with a substantial component devoted to the subject; now there are at least half a dozen. There were few conferences; now there are at least five, every year, in locations ranging from Korea to the U.S.A., Mexico, Austria and France. Thirty-five years ago there were only a handful of universities where a graduate student could receive training in several complex variables; now there are several dozen. Section 32 of the AMS Subject Classification Scheme expanded considerably in the last revision of the Classification; and the number of papers being written in that area has expanded in the past four decades by a factor of at least twenty.

The theory of several complex variables is a growth area for mathematics. Both a source of challenging problems and of powerful techniques, it continues to inspire hundreds of mathematicians and to create currents of activity worldwide. Part of the excitement of several complex variables is its unpredictability: we never know what technique or what other part of mathematics will become relevant to the problem at hand. We all look forward to an exciting period of new explorations in the subject.