Unlikely intersections in algebraic groups and Shimura varieties

organized by
Jonathan Pila and Umberto Zannier

Workshop Summary

This workshop, co-sponsored by AIM, NSF, and the Centro di Ricerca Matematica Ennio De Giorgi of the Scuola Normale Superiore di Pisa, focused on the diophantine conjectures on "unlikely intersections" formulated by Zilber, Pink, and Bombieri-Masser-Zannier. These conjectures give a far-reaching generalization of the Mordell-Lang and André-Oort conjectures and are the subject of much current work. They are connected with classical diophantine problems for semi-abelian varieties, the arithmetic of Shimura varieties, and with model theory – indeed they were independently formulated from these three perspectives. Methods and ideas from model-theory have also proved useful in achieving various partial results. The primary purpose of the workshop was to bring together researchers from these three disciplines with a common interest in the problems. Additionally, researchers in arithmetic dynamics were invited as the problems have natural analogues there, and model-theoretic methods are also used in that area.

The morning sessions were devoted to lectures. Early in the week these were of an introductory nature, aimed at introducing the different communities and their perspectives to each other. On Monday: Enrico Bombieri and Boris Zilber explained their perspectives on the conjectures. On Tuesday, David Masser discussed his work with Zannier on some problems of "relative Manin-Mumford" type that are encompassed by the general conjectures, and Andrei Yafaev spoke about the André-Oort conjecture and its resolution under GRH in his work with Klingler and Yafaev. On Wednesday, Lucien Szpiro gave an introductory talk on arithmetic dynamics. Subsequent talks were devoted to more specialised tools and results. A second talk on Wednesday, by Daniel Bertrand with contributions by Bas Edixhoven and Shou-Wu Zhang, described a subtlety in the conjectures that leads to a counterexample of a natural (but insufficiently careful) formulation of Pink's "relative Manin-Mumford conjecture". On Thursday, Kobi Peterzil gave a talk that was in part an introduction to o-minimality and in part a description of his work with Starchenko on definability of theta functions. Jonathan Pila described his unconditional proof of André-Oort for products of modular curves relying on o-minimality and his counting result with Alex Wilkie. On Friday Angus Macintyre talked about the model theory of exponentiation, especially comparing the classical and Zilber exponential. Paula Tretkoff spoke about transcendence theory particularly regarding the uniformising functions of Shimura varieties.

Afternoons were devoted to group sessions. On Monday and Wednesday these were sessions in which an expert (in model theory, number theory or Shimura varieties) answered questions from those in other areas. There were two parallel sessions, essentially along a number theory - model theory division. Topics discussed included: heights, Shimura varieties, basic model-theory and o-minimality. The "experts", Philipp Habgger, Jonathan

Kirby, Thomas Scanlon, Sergei Starchenko and Bas Edixhoven, all did an excellent job answering diverse questions with thorough explanations.

The remaining afternoons were devoted to problem sessions. Problems were retained for further work where there was continuing interest, and additional problems were also contributed along the way. The initial problem session was moderated by Philipp Habegger. Reports of the discussions were then made before each ensuing session. Problems discussed included (this list is not exhaustive):

* Questions about diophantine properties of definable sets. In particular, the following question was raised by Jacob Tsimerman. *Let X be a semi-algebraic set whose integer points $X(\mathbb{Z})$ are Zariski dense. Suppose that Y is definable in an o-minimal structure and contains $X(\mathbb{Z})$. Is $\dim Y \geq \dim X$?* A counterexample was constructed by Enrico Bombieri (who was not in that group but was told about the problem en route to the Banquet) using Matijasevich's theorem.

* Model theory of a field with a height function. Heights and their properties are crucial to most work on the diophantine problems. Can they be incorporated into the model-theoretic picture? This problem was pursued over 3 afternoons, with various formulations.

* Are there only finitely many singular moduli which are algebraic *units*? This is not part of the conjectural framework, but a more general "unlikely occurence" conjecture. This was proved during the week by Philipp Habegger. Variants and generalisations were also raised.

* Describe all rational triples a, b, c such that there exists singular moduli j_1, j_2 with $aj_1 + bj_2 + c = 0$.

* An analogue "unlikely intersection" problem for varieties over a field intersected with varieties (of codimension exceeding the dimension of the given variety) over a smaller field. A positive solution was found by the group working on this.

* To find explicit bounds for the number of connected components of certain transcendental analytic sets. Some initial specific problems were solved.

* A model-theoretic framework for conjectures in algebraic dynamics.

Overall, the workshop provided an opportunity for a sustained interaction between model-theorists, number-theorists and arithmetic geometers enabling an exchange of ideas at a level of detail as well as in the large. Several participants commented that they enjoyed the working sessions, some of which were intensively pursued over 2 or 3 afternoons. Altogether it was an instructive and enjoyable meeting, hosted in the beautiful surroundings – and with the excellent organisation – of the Centro de Giorgi, and with careful guidance on the structure of the sessions by the AIM representatives. We thank all these people and institutions for supporting and contributing to the success of this workshop.