Triangulations, Heegaard splittings and hyperbolic geometry

organized by
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Workshop Summary

The workshop activities focused on an array of themes, some expected, some less expected, and some peripheral, though tantalizing. From the expected themes, two central projects emerged: 1) Understanding triangulations and their relation to geometry in general and Mom structures in particular; 2) Understanding Namazi's construction of a hyperbolic metric from a high distance Heegaard splitting.

During a sequence of group discussions devoted to triangulations, participants asked and discussed several pertinent questions: Are there topological criteria for deciding whether or not a triangulation is canonical or geometric? How does SnapPea choose and rechoose triangulations? How do efficient triangulations inform our understanding of Mom structures of small complexity? Is there a Ricci-free proof of the Geometrization Conjecture? Recent work of Guéritaud and (less recent) work of M. Kapovich suggest that this is possible, at least in certain cases.

Namazi led a discussion of his construction of a hyperbolic metric from a high distance Heegaard splitting. His construction is intricate and ties into several branches of mathematics only peripherally related to the study of 3-manifolds. Though this prevented him from providing full details for his construction, it did not prevent the participants from active questioning and thereby a chance of understanding the flow of the argument.

Several interesting projects emerged from the less expected themes, most notably the following: 1) Understanding and generalizing the tree of tunnel number 1 knots constructed by Cho-McCullough; 2) Verifying recent advances on the stabilization problem; 3) Understanding the behaviour of Heegaard genus under Dehn filling.

Interest in the work of Cho-McCullough turned out to be overwhelming and a lecture was scheduled on this topic. (It was given by Scharlemann, who stood in for McCullough, who could not attend due to inclement weather.) The construction relies on a Theorem due to Georitz that holds for Heegaard splittings of genus 2 of $\mathbf{S}^3$. Group discussions on this topic asked whether or not Goeritz' Theorem can be extended to higher genus Heegaard splittings of $\mathbf{S}^3$ and on alternate descriptions of the edges in the tree of tunnel number 1 knots.

In a sequence of group discussion, Bachman presented an example of a 3-manifold with distinct Heegaard splittings, one of genus g and one of genus g', that are likely to be equivalent only after $g + g' - 1$ stabilizations. His argument relies heavily on his theory of sequences of generalized Heegaard splittings. The sequence of group discussions made some progress examining the details of the underlying theory, though not enough to reach a consensus at this time. Bachman's announcement precipitated a similar announcement by Hass-Thompson-Thurston and Hass presented a compelling outline of the argument. In essence, Hass-Thompson-Thurston construct a 3-manifold M with special properties (in a very rough sense, its geometry is "long and thin") that imply that a Heegaard splitting $M = V \cup_S W$ of genus g must be stabilized at least g times before becoming equivalent to (the appropriate stabilization of) $M = W \cup_S V$. The key ingredient lies in considering harmonic maps to this 3-manifold.

There was a lively discussion on the behaviour of Heegaard genus under Dehn filling. This ties in with several classical questions of interest to 3-manifold topologists. Many questions were put forward, some of which were answered on the spot. Several of those that were not are included in the list of open problems.

Some very creative work occurred in less central group discussions. One such group tried to integrate an understanding of Heegaard Floer homology with more classical results on 3-manifolds. This group found that some of the classical results in sutured manifold theory admit slick modern proofs via Heegaard Floer homology.

Another group considered notions of randomness for 3-manifolds. The most standard such notion relies on random walks in the Cayley graph of the mapping class group of a surface of genus g and identifies two genus g handlebodies according to a random element of the mapping class group thus obtained. Moriah presented an alternate notion of randomness. Both notions imply that, in a certain sense, "most" 3-manifolds are hyperbolic. (In the first case, this follows from work of Maher and the fact that high distance Heegaard splittings are essentially unique, in the second case, it follows from a recent theorem of Moriah-Lustig.) The group expressed concern over the bias in the two notions of randomness. (Both work with a fixed genus g. For some questions, this is immaterial, not so in others.) Licata suggested a new notion of randomness for knots and links: Consider integral points in a cube of size $n \times n \times n$ (vertices), add in all line segments of length one connecting two such points (edges). In the resulting graph, resolve crossings at the vertices according to a simple probabilistic procedure (there is more than one option for such a procedure). One advantage of this notion over considering random closed self-avoiding walks is that there is no bias in the number of components of the resulting (knot or) link.

Overall, the workshop was a great success. Many participants lauded the organizers and thanked them for one of the best events they had ever attended.