The Tate conjecture

organized by
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Workshop Summary

The purpose of the conference was to bring experts together in various aspects of the Tate conjecture to synthesize the various approaches and prepare the ground for further advances. Since the conjecture was posed in 1963, there have been periods of great activity followed by more fallow periods. The first active period was in the mid 1960's, when Tate, himself, proved that for an abelian variety A over a finite field $\mathbb{F}$ of characteristic p and a prime number $\ell \neq p$, the natural map:

$$\mathrm{End}(A) \otimes \mathbb{Q}_\ell \to \mathrm{End}_{G_\mathbb{F}}(V_\ell(A)),$$

is bijective, where $G_\mathbb{F}$ is the absolute Galois group of $\mathbb{F}$ and V_ℓ is the ℓ-Tate vector space. In the early 70's, Zarhin proved this statement for abelian varieties over most function fields over a finite field, and in 1983, Faltings proved it over any finitely generated field. There were results for K3-surfaces of finite height over finite fields due to Nygaard and Ogus, and results for Shimura varieties, due to people such as Harder-Langlands-Rapaport, Klingenberg, Ramakrishnan and Murty, and Blasius. Also, there are several nice results of the form "Hodge implies Tate" or "Tate implies Hodge". For example, Milne proved that the Hodge conjecture (in any codimension) for CM abelian varieties over $\mathbb{C}$ implies the Tate conjecture in any codimension over finite fields, and Deligne proved that the Tate conjecture for abelian varieties over finitely generated fields implies the Hodge conjecture over $\mathbb{C}$.

There were 10 one-hour lectures at the meeting, two per day, and a few shorter, more informal talks. By Tuesday afternoon, the participants had enough background to begin to break up into groups to work on specific problems and learn about some aspects of the subject in more detail. Since there was a mix of experts and less experienced mathematicians, the first few talks of the conference were very general and helped get the participants up to speed. There are many basic results that have not been adequately recorded, and it was very helpful to the participants and the organizers to have them summarized by experts.

One interesting aspect of the conference was the presence of a columnist for *The Wall Street Journal*, Lee Gomes. He bravely attended most of the talks in the first three days of the conference and interviewed several participants extensively. He then wrote an article on the conference that appeared on August 1. It is not often that mathematicians' names are to be found in *The Wall Street Journal*, so this was considered a coup.

Our aim was to have experts in various aspects of the subject get together and share their ideas, with the hope and expectation that this will lead to cross-fertilization and even new methods. We think it is fair so say that this goal was realized quite well.

The following are some of the general themes that were explored in the lectures and working sessions:

(1) The Hodge and Tate conjectures for abelian varieties: Raskind outlined some of the known results for divisors on and endomorphisms of abelian varieties and Milne gave some indications of one of the basic methods of proof used by Tate, Zarhin and Faltings. Milne then explained his approach for proving that the Hodge conjecture for CM abelian varieties over $\mathbb{C}$ implies the Tate conjecture over a finite field. Milne and Izadi outlined some of what is know about the Hodge conjecture for abelian varieties over $\mathbb{C}$, in particular exotic classes (not in the subring of the Hodge ring generated by rational (1,1) classes) and when they are known to be algebraic. Somewhat related is the case of K3 surfaces, due to the Kuga-Satake construction which associates an abelian variety A of dimension 2^{19} to the primitive cohomology of a K3 surface X over $\mathbb{C}$ using the even Clifford algebra, and Deligne's construction of an absolute Hodge cycle on $X \times A$, by means of which one may prove the Tate conjecture for divisors on K3 surfaces over a field of characteristic 0. This does not work in positive characteristic, and Zarhin outlined how one can prove Tate for K3 surfaces of finite height over finite fields, and on products of some such surfaces.

(2) The theory of Shimura varieties provides us with many examples of Hodge and Tate classes. This was addressed in the talks of Ramakrishnan and Murty. In fact their work matches in some situations the Hodge and Tate cycles by period relations involving appropriate L-values, which is then used to deduce the Tate conjecture for divisors on a class of Shimura surfaces from the Hodge conjecture, which is known by Lefschetz. Despite knowing the Tate classes to be algebraic, we often don't know explicit algebraic representatives for them. This is the case for Hilbert modular surfaces over a non-abelian extension of $\mathbb{Q}$. If we knew better such representatives, we could intersect them with modular curves and possibly define new interesting points on such curves. There are also natural Tate classes on the product of two Hilbert modular surfaces coming from different real quadratic fields. How can one find algebraic representatives for these? In general, much less is known about Hilbert-Blumenthal surfaces in positive characteristic, where it seems that "special" cycles (e.g. Hirzebruch-Zagier cycles and their Hecke translates) are not enough to account for all of the Tate classes.

(3) The Tate conjecture can only be expected to be true with $\mathbb{Q}_\ell$-coefficients. But for a given X, can we expect it to be true on the $\mathbb{Z}_\ell$-level for almost all ℓ (we think so)? Over finite fields, this question is equivalent to discerning the difference between Weil étale cohomology and the usual étale cohomology. For dimension 1-cycles, Schoen has shown that the rational Tate conjecture implies the integral Tate conjecture. We will have non-surjectivity on the $\mathbb{Z}_\ell$-level for 0-cycles if ℓ divides the minimal degree of a zero-cycle on X.

(4) Izadi discussed results on the Hodge conjecture for some complex varieties and how one might hope to transport some of the techniques to prove the Tate conjecture in

some cases. Can Lefschetz's original idea of proof of his famous (1,1)-theorem (as refined by Griffiths) be adapted to prove the Tate conjecture for divisors, at least in some cases?

(5) Ulmer described his technique for constructing elliptic curves of large Mordell-Weil rank over function fields over finite fields, using the Tate conjecture for certain surfaces over finite fields. In some of the working sessions, he discussed attempts to adapt this technique to certain function fields over number fields, and the related problem of whether certain curves over number fields have Jacobians with large endomorphism algebras.

(6) There is a local version of the Tate conjecture for varieties with totally degenerate reduction. If the variety comes by base change from a variety over a number field, then the local Tate conjecture implies the usual Tate conjecture. One good test of this conjecture would be to prove that for an abelian variety with totally multiplicative reduction over a p-adic field, K (finite extension of $\mathbb{Q}_p$), the natural map

$$\mathrm{End}(A) \otimes \mathbb{Q}_p \to \mathrm{End}_{G_K}(V_p(A))$$

is bijective.

(7) The Tate conjecture for varieties over finite fields has some strong implications, such as that numerical and rational equivalence of cycles coincide, up to torsion, Parshin's conjecture that higher algebraic K-groups should be torsion, and to conjectures on values of zeta-functions and their relationship with étale, motivic and Weil étale cohomology. This was discussed in the talks of Lichtenbaum and Spiess.

(8) Flach discussed the problem of how to define Weil étale cohomology for rings of integers in number fields and, more generally, for arithmetic schemes, and to prove that it has all of the desired properties. This would help us go from the Tate conjecture to precise statements about values of zeta and L-functions for varieties over number fields.