Representations of surface groups

organized by
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Workshop Summary

The workshop met at the American Institute of Mathematics March 19-23 2007. Each morning there were two expository talks for a general audience. After a long lunch break there were 2-4 informal discussion groups, often based on the morning's general talks. The discussion groups reported to the whole audience at the end of each afternoon.

This workshop followed several previous conferences on surface group representations, but was different in several respects. The previous meetings focused on the relations between Higgs bundle moduli and bounded cohomological methods for representations on Hermitian symmetric spaces. While this aspect continued to play an important role, the subject matter of the AIM workshop ranged more widely than this. The general talks touched on topics including classical Kleinian groups ($\mathsf{SL}(2,\mathbb{C})$) and hyperbolic 3-manifolds, the Thurston theory of geodesic laminations on hyperbolic surfaces, and the relation between surface group representations and signatures of surface bundles over surfaces.

The participants came from several different research areas. One purpose of the workshop was to foster interaction between the various groups, and we felt that the workshop met this goal. With so many participants from different specialties, the organizers felt that the beginning of the workshop should concentrate on establishing a common language and background. The preliminary documents — an introductory paper of over forty pages, and a glossary composed by the participants — were meant to achieve this goal. By the end of the second day, after the basic introductory lectures, the workshop took on a more concentrated focus.

Specific topics discussed in the afternoon sessions included the following:

1. Geometric interpretation of maximal components of representations in $\mathsf{Sp}(4,\mathbb{R})$. Peter Gothen, in his thesis, counted the maximal components using Higgs bundle techniques. (In fact he presented this in one of the morning lectures.) Some of them, such as the *Hitchin components,* could be identified as containing deformations of Fuchsian representations into $\mathsf{SL}(2,\mathbb{R})$. A natural question, of great interest to the four organizers, is whether *every maximal representation* into $\mathsf{Sp}(4,\mathbb{R})$ can be deformed into a strictly smaller algebraic subgroup. (The analogous question for unitary groups $\mathsf{U}(p,q)$ is known to be true.) This question stimulated intense interest for the last few days of the workshop. While the question was unresolved, new explicit examples of maximal representations in $\mathsf{Sp}(4,\mathbb{R})$ were constructed, and we have proposed conjectures about how they fit into the picture obtained from Gothen's calculation.

2. Higher Teichmüller theory Another subject which attracted intense interest is the recent work by Fock and Goncharov on "higher Teichmüller theory" (where "lower Teichmüller theory" would be the study of hyperbolic structures on compact surfaces, as

pioneered by Fricke and Klein in the century preceding Teichmüller). This subject starts from laminations on a closed surface ("ideal triangulations") and associates parameters describing how ideal polygons are glued to form the surface. The Poisson structure on the moduli spaces admits a particularly simple form, and the mapping class group action extends to a groupoid generated by elementary operations on ideal triangulations, called *flips*. Using these parameterizations, there had been some very recent advances in this area, generalizing important identities for lengths of geodesics on hyperbolic surface. Many of the participants had heard of this theory, and the workshop provided a useful introduction to this subject. In particular relating their synthetic geometry approach to the more analytic approaches using harmonic mappings and Higgs bundles over Riemann surfaces appeared as a central and quite difficult problem.

3. Invariant complex structures A related, but different problem, which also received considerable interest during the first days of the workshop, is to define mapping class group invariant complex structures on Hitchin components. One approach to this is to minimize the energy functional. This technique has been discussed in detail in the case of $\mathsf{SL}(3,\mathbb{R})$, were it actually gives a solution.

4. Boundary maps Another topic which attracted much interest, and seems destined to a fundamental role in this subject, concerns the boundary values of twisted harmonic maps and their relation to the measurable equivariant mappings from $S^1_\infty = \partial\pi_1(\Sigma)$ to the Furstenberg boundary. The continuity of the Furstenberg map is a key step in the work of Burger-Iozzi-Wienhard, as well as the rigidity theorems of Mostow, Margulis, et. al., which were predecessors to this work. The conditions under which the Furstenberg map extends continuously and agrees with the boundary values of the harmonic sections received much attention at the workshop.

The workshop ended with a discussion session on open problems and a final session on future plans.

As was announced at the final session of the workshop, a week-long workshop at Luminy on several of these questions is being planned for September 2008. The focus of this workshop will be similar to the AIM workshop. It may involve several other related topics which, for reasons of space and time, could not be discussed in detail at AIM. These include mapping class group actions on moduli spaces, translation surfaces in the moduli space, and other geometric structures (hyperbolic structures and Lorentzian structures). The vitality of the subject covered by the AIM workshop was evident in the way the focus of the AIM workshop differed from the preceding workshops in Basel-Strasbourg and Madrid. We expect that the workshop in Luminy will represent yet another evolution in this rapidly developing subject.

Based on material of the AIM workshop we plan to set up a webpage (managed by one of the participants, Graeme Wilkin). In addition to a list of open problems based, initially, on the workshop problem session, the webpage will include relevant background and research material, a wiki-style glossary, and will provide a medium for discussion and interaction between the different groups interested in surface group representations.

We thank the AIM faculty and staff for their expert help in making the workshop a memorable and rewarding experience.