Supercharacters and combinatorial Hopf algebras

organized by
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Workshop Summary

The primary goal of the this workshop was to establish a line of communication between researchers in the emerging area of supercharacter theory and experts in combinatorial Hopf algebras. We decided to focus on the following problem which arose from the representation theory of the group of upper triangular matrices with entries in a finite field. In the theory of "super characters and super classes" the natural objects are all indexed by set partitions. There is another actively studied object with a basis indexed by set partitions, the symmetric functions in non-commuting variables. Several of us felt that there might be a connection between these two subjects. This became:

Problem. *Is there a Hopf algebraic connection between the Hopf algebra of symmetric functions in non-commuting variables and the ring of supercharacters of the unipotent upper-triangular matrix groups over a finite field with two elements?*

At the time of the workshop, it was known that there were many algebra isomorphisms between these two spaces, but it was very unclear whether a corresponding Hopf structure isomorphism was possible.

Our workshop not only succeeded in discovering an explicit isomorphism between the two spaces, but also in tackling the following variations:

- A Hopf algebra isomorphism between the ring of supercharacters of the unipotent upper-triangular matrix groups over an arbitrary finite field and a Hopf algebra of symmetric functions in colored non-commuting variables.
- A symmetric function realization for the Hopf algebra dual space of the ring of supercharacters of the unipotent upper-triangular matrix groups over an arbitrary finite field.

The participants have now written a preprint "Supercharacters, symmetric functions in non-commuting variables, and related Hopf algebras" (arXiv:1009.4134v1), detailing the results of the workshop.

Workshop details

We began the workshop with two survey talks: Nat Thiem gave an introduction to supercharacters with an emphasis on the case of unipotent upper-triangular matrices, and Nantel Bergeron gave an introduction to combinatorial Hopf algebras with an emphasis on the Hopf algebra of symmetric functions in non-commuting variables. That afternoon was spent in discussion groups hoping to address confusions from the morning talks in greater detail. It also marked the start of project, spearheaded by Franco Saliola, of implementing the Hopf relations into SAGE.

The second morning's talks gave some additional background material: Mike Zabrocki described the classical connection between the characters of the symmetric group and the Hopf algebra of symmetric functions (which was to serve as a model for our problem), and Marcelo Aguiar broadened the picture to the slightly more general setting of Hopf monoids. The afternoon was dedicated to actually begin understanding the combinatorial underpinnings of the two structures we were studying, and – to our surprise – we found an answer to our problem in the course of these discussions.

With this success under our belt, we decided to broaden our scope. To facilitate this goal, we had three more general talks on the third morning: Jean-Yves Thibon gave a more general talk about combinatorial Hopf algebras, Carlos André described how some of the nice features of the supercharacter theory of our main example generalize to other settings, and Anders Henrickson discussed some ways to construct new supercharacter theories from existing ones. As our isomorphism from the previous day had raised more questions than it had answered, we spent the afternoon in groups tackling some of these different questions. This afternoon we managed to not only find the above-mentioned variations to our problem, but we also found some more negative results, such as the fact that our supercharacters were not Schur positive under this isomorphism.

On the fourth day, Persi Diaconis began with a talk to address a concern that had been nagging many of the participants throughout the workshop: It is well-known that the usual character theory of the unipotent upper-triangular matrices over a finite field is wild, but most participants only had a vague notion of what "wild" means mathematically. With this clarified, Kay Magaard gave a talk about work with Tung Le on decomposing supercharacters into more irreducible characters. Nantel Bergeron finished off the morning with a talk clarifying the meaning of the antipode map. The afternoon was spent beginning to think about problems that would not be realistically solved in the course of the workshop. Some problems that were considered included: what are other nice supercharacter theories for the group of unipotent upper-triangular matrices, and can one glue together supercharacter theories of the symmetric group to get a combinatorial Hopf algebra related to symmetric functions in some way?

The final day of the workshop was largely focused on establishing what we had done and looking to the future. Eric Marberg described other towers of groups which have a combinatorial supercharacter theory description, and Franco Saliola gave an introduction to the use and procurement of the new SAGE package. Then the focus turned towards planning next moves. We decided to write a joint paper among the participants cataloguing our results, and in the afternoon we listed all the open problems we had discovered over the course of the workshop.

Outcomes

We had a wide range of participants, from first year grad students to senior faculty, and all seemed to find some niche to actively contribute to the overall project. Participants coming from either side of the initial divide (supercharacter and Hopf algebras) learned a great deal about both subject areas. Our biggest contribution seems to be the idea that one can give a representation theoretic underpinning to a combinatorial Hopf algebra by considering supercharacter theories in addition to the usual representation theories.

Tangible products developed by the workshop include the following.

- A workshop preprint which is submitted for publication.
- An implementation of the supercharacter theory of unipotent upper-triangular matrices over a finite field into SAGE, now available in the SAGE combinat package.
- An implementation of the supercharacter theory of unipotent upper-triangular matrices over a finite field into muPad.
- A list of open problems.