CRITICALITY AND STOCHASTICITY IN QUASILINEAR FLUID SYSTEMS

organized by
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Workshop Summary

Report: AIM Workshop on Criticality and Stochasticity in Quasilinear Fluid Systems

Overview

In recent years, several major breakthroughs were made in the field of mathematical fluid dynamics, and several of these topics played a central role in our workshop, "Criticality and Stochasticity in Quasilinear Fluid Systems (part 2)."

Lectures and Sessions

The 5-days workshop ran everyday with two talks each morning:

(1) Monday 05/02: Marc Brachet and Dallas Albritton
(2) Tuesday 05/03: Bartosz Protas and Susan Friedlander
(3) Wednesday 05/04: Roman Shvydkoy and Alex Kiselev
(4) Thursday 05/05: Gautam Iyer and Vincent Martinez
(5) Friday 05/06: Wojciech Ozanski and Mikhail Vishik

In the afternoon sessions and private discussions, we made significant progress to these open problems and more. For many participants, this AIM workshop was the first in-person meeting in several years and everybody really appreciated deep dialogues and exchanging of views.

Open Problems and Discussions

Here, we summarize various highlights of the discussions in the workshop, as well as some of the proposed open problems.

Non-uniqueness and Singularity Formation.
Arguably the most important equations in fluid dynamics, the Navier-Stokes equations and the Euler equations, have fundamental questions about their solutions which remain unanswered. In particular, the basic question of whether the solutions can "blow up" (i.e., develop a singularity), or whether they are "globally well-posed" (no singularity can form) remains a challenging open problem. This problem is so important and difficult that its resolutions has a famous $1,000,000 prize attached to it (see [Fefferman2000Clay]). Another important question is, if the solutions blow up in finite time, what would be the appropriate functional space in which a unique solution exists? In our workshop, we had several talks and much

discussion revolving around these problems, involving both analytical and computational questions, some of which is described below.

Analytical investigations. The Navier-Stokes equations were recently shown in [albritton2022non] to have solutions that are "physical" in the sense that they obey what is known as the Leray-Hopf energy inequality, but which lose uniqueness even for the same initial state, putting the very notion of predictability in jeopardy. Dallas Albriton, one of the authors of [albritton2022non], was also one of the speakers at the workshop. His talk inspired much lively discussion and examined many open questions. Moreover, the ideas of [albritton2022non] built upon the work [vishik2018instability2,vishik2018instability1] of another workshop participant, Prof. Mikhail Vishik, who also gave an inspiring talk which generated much discussion. To witness the exhilarating scientific discussions between Albriton, a rising star at the beginning of his career, and Vishik, a veteran of fluid dynamics and a legend in the eyes of many, was a treat for all to behold, and led to many new ideas, some of which are discussed below. In particular, [albritton2022non] used what is known as a forcing function with a singularity, which some consider somewhat non-physical. The race is now on to see if these results can be extended to the case of more physical forces.

(1) Vincent Martinez gave a talk on unique ergodicity, regularity, and mixing for the damped-driven stochastic KdV equation and suggested the following further problems.
(2) Can one prove uniqueness of an invariant measure for two-dimensional stochastic Euler equations with Ekman friction (damping) term and additive noise? In short, the lack of diffusion makes the standard approach difficult to go through.
(3) Can one provide examples (other than the stochastic one-dimensional damped cubic nonlinear Schrodinger equation) of weakly damped systems where relaxation to the unique invariant measure occurs at an algebraic rate, rather than exponential?
(4) It would be highly worthwhile to gain better understanding of Kuksin measures and its construction in general.

Computational investigations. In the numerical context, both Marc Brachet and Bartosz Protas presented in-depth computational studies related to the problem of singularity formation for the Euler and Navier-Stokes problems. While many topics were discussed, major themes involved trying to understand what kinds of constraints numerical observations appear to impose potential analytical results, and what computational results can potentially tell us about the equations. For example, a phenomenon known as *thermalization* occurs in the 1D Burgers equation, which for many years was dismissed as just numerical noise, may actually be telling us something about resonances related to singularity formation. It is an open problem to see how far these ideas can be pushed into the Navier-Stokes or Euler context. In addition to this, the talk of Protas focused on computationally probing the sharpness of analytical estimates. Via a series of careful and computationally intensive optimization calculations, strong evidence was presented that the well-known analytical bounds for these equations appear to be sharp, indicating that either singularity formation is possible, or that if future analysts are to prove that no singularity can form, then new techniques will be needed.

We note also that, in the course of these discussions, Dallas Albritton, Marc Brachet, and others talked about bifurcations in the problem of flow past a cylinder. In particular, there is currently no rigorous proof that a bifurcation happens as the Reynolds number increases, something that has long been observed in experiments and simulations. Marc Brachet suggested to look at the analogous problem for the Gross-Pitaevskii equation (GPE), for which somewhat more is known. It was suggested that the GPE version of this problem may therefore be more amenable to rigorous proof, and may be a starting place for tackling the Navier-Stokes case.

Many interesting open questions arose, including the following.

- What kind of singularity diagnostic should be used in the Euler and Navier-Stokes equations?
- Has thermalization in spectrally truncated PDEs any usefulness in the singularity context?
- Could Gross-Pitaevskii superfluid dynamics be relevant in a classic context?
- Rigorous derivation of a Hopf bifurcation in Reynolds number for 2D incompressible Navier-Stokes flow or Gross-Pitaevskii flow passing a (cylindrical) obstacle. Can it be proven without computer-assistant proof?
- Prove sharp estimates of the H^1 norm of solutions to the 1D periodic viscous Burger's equation. It is proven the enstrophy $\mathcal{E}(t) := \|\partial_x u(t)\|_{L^2}^2$ obeys $\mathcal{E}'(t) \leq C\mathcal{E}(t)^{5/3}$. Prove the following bound which comes from numerical evidences: $\mathcal{E}(t) \leq C\mathcal{E}(0)^{3/2}$
- Study the bifurcation problem of Taylor-Couette instability.
- In 2D, formulate the random vortex methods analog to the model of a flow past an obstacle (Stochastic ODE). Is there a stability transition (Hopf bifurcation) in the Reynolds number?
$$\frac{dz_i}{t} = \sum_{j\neq i} v(z_j) + u_{\text{background}}$$
where $v(z_i)$ is the velocity generated by a point vorticy $\delta_{z=z_i}$.
- Study the inviscid limit of the non-unique Leray solution to the forced Navier-Stokes equation.

Multi-Physics.
Criticality and stochasticity arise in many areas with close relations to quasilinear fluid systems. For example, in our workshop Susan Friedlander discussed equations for magnetic relaxation, and how they might tell us about steady states to the Euler equations. Roman Shvydkoy discussed equations that describe the flocking of birds and other animals, which bear a striking resemblance to the equations of fluids. Alex Kiselev discussed relationships between the Euler equations and the Surface-Quasigeostrophic (SQG) equations, including situations in which the initial data is given by a “patch”-like function. Wojciech Ozanski talked about interactions between fluids and structions (FSI).

- Prove the global existence of (strong) solutions to the 2D Burgers' equation with hyperviscosity; namely, $\partial_t u + u \cdot \nabla u = -\nu \triangle^2 u$. This problem has been known about since at least the publication of [LariosTiti2015BCBlowup]. While the problem seems straight-forward, since it appears to be merely Burgers equation with more diffusion, this is far from the case. Indeed, the presence of the bi-Laplacian in the hyperdiffusion term means that the maximum principle no longer holds, killing any hope of extending the usual techniques used to prove global well-posedness. While several ideas were discussed, it seems that a resolution of this problem may be out of reach with current techniques.

- Susan Friedlander proposed to investigate the global asymptotic behavior of the 3D Magnetic relaxation equations (MRE):

$$\begin{cases} \partial_t B + u \cdot \nabla B = B \cdot \nabla u, \\ (-\triangle)^\gamma u = B \cdot \nabla B + \nabla p, \\ \nabla \cdot u = 0, \end{cases}$$

  with connections to the Magnetohydrodynamics (MHD) equation and the Euler equation. This problem, first looked at in detail by K. Moffatt, may provide insights into steady-states of the Euler equations. Indeed, if the large-time dynamics of these equations are trivial (e.g., constant), then one automatically obtains a Euler steady state by passing to the infinite-time limit. During the workshop, A. Larios coded a solver for this equation, and ran preliminary solutions, which seem to indicate that the large-time dynamics are indeed constant; a promising step toward deeper understanding of this problem.
- Wojciech Ozanski discussed the the 1D surface growth model, given by

$$\partial_t u + u_{xxxx} + \partial_{xx}(u_x^2) = 0$$

  He proposed to prove (or disprove) the global existence of strong solutions to the inviscid version of this model.
- Does the Euler Alignment System for flocking behavior have blow up, or global well-posedness?

Mixing.
A major topic in the field of fluid dynamics is the problem of mixing; namely, one wishes to quantify, understand, and predict the common experience of a substance (e.g., colored dye, milk in coffee, etc.) spreading through a fluid via the swirling motion of the fluid combined with diffusion processes. Many difficult, practical problems in this area remain open. Indeed, even the problem of deciding on a measure of how "mixed" a fluid solution is at a given time is not a fully resolved problem, though many measures have been proposed, such as the H^{-1} norm of tracer particles in the fluid.

Both Gautam Iyer and Vincent Martinez gave intriguing talks on topics related to mixing.

Martinez's talk focused on the idea of using a technique from the field of data assimilation known as "nudging", specifically the Azouani-Olson-Titi method. In particular, it was noted that large-scale nudging as a control strategy is a paradigm for deducing unique ergodicity in spite of weak dissipation. Moreover, the nudging paradigm clarifies some of the main ideas, even leading to simplified arguments for some deterministic results. Some of these ideas were extended to the stochastic setting, which is highly relevant to mixing. In particular, Martinez showed that the structure of conservation laws for the Korteweg-De Vries (KdV) equation allow one to bootstrap estimates even in the stochastic setting. In addition, we learned that algebraic moments suffice for unique ergodicity, exponential moments suffice for exponential mixing, which raises a natural question (pointed out by Martinez): Do algebraic moments suffice for algebraic mixing?

- relations to nudging and data assimilation, same approach holds for damped-driven stochastic NLS and mKdV, alternative approach via Strong Feller + irreducibility, Degenerate noise case,

- Can one show polynomial mixing rates occur in the case of small damping?
- Can the techniques applied for KdV be extended to other disperse equations such as Benjamin-Ono?
- How much of this program can be extended to the case of so-called "'weak damping" (i.e., negative powers on Laplacian), and what, if anything, can be said about the zero-damping limit?
- Do algebraic moments suffice for algebraic mixing?
- Gautam Iyer proposed to investigate the Doering Conjecture: Given a velocity field u for the transport-diffusion equation
$$\partial_t \theta + u \cdot \nabla \theta = \kappa \triangle \theta$$
show that
$$\frac{\|\theta\|_{H^{-1}}}{\|\theta\|_{L^2}} \to C \text{ as } t \to \infty.$$
This problem was discussed in small-group format in the afternoon sessions with promising progress started before the end of the workshop.

One of the organizers, Yamazaki, also discussed with Martinez about the possibility of considering the KdV equation forced by space-time white noise which will require applications of different techniques such as the theory of paracontrolled distributions and regularity structures.

Conclusion

Many new results and open problems were discussed in this workshop, and many new research collaborations were forged. We feel that this workshop was a great success, and will lead to many new results in the coming months and years.

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