Set theory and C*-algebras

organized by
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Workshop Summary

This workshop was devoted to applications of set theory to C*-algebras, with the main theme being classification. Both C*-algebras and their classifying invariants (such as the Elliott invariant or the Cuntz semigroup) can be naturally parametrized as standard Borel spaces. The computation of invariants is also expressed in terms of Borel functions. These facts make the classification problem for C*-algebras subject to set-theoretic methods, and negative results about the Borel reducibility of the isomorphism relation on C*-algebras to some benchmark equivalence relations (such as isomorphism of countable structures or orbit equivalence relations of Polish group actions) present the ultimate negative classification results. One of the original co-organizers of the workshop, Greg Hjorth, died suddenly in January 2011. His work, and in particular the notion of a turbulent group action, played a key role in many of the discussions during the workshop.

The morning sessions of the workshop were reserved for lectures, some of which were prompted by discussions and questions during the early part of the week. On Monday morning, talks of Ilijas Farah and David Kerr mapped out two of the main directions of the workshop. With an eye towards developing novel applications of combinatorial set theory to the study of phenomena in nonseparable C*-algebras, Farah gave an introduction to Naimark's problem of whether the unitary equivalence of all irreducible representations of a given C*-algebra A is equivalent to A being isomorphic the compact operators on some Hilbert space. As was shown several years ago by Akemann and Weaver, the existence of a counterexample to Naimark's problem is a consequence of $\diamondsuit$ and hence is consistent with ZFC, but it is an open question whether a positive answer is consistent with ZFC. Within the realm of descriptive set theory, Kerr discussed Borel reducibility as basis for analyzing the complexity classification problems and outlined the role of Hjorth's dynamical concept of turbulence as an important tool in this program. Some indications were given as to how these ideas might apply in a noncommutative C*-algebraic framework, for example in the study of automorphism groups.

On Tuesday morning, Mikael Rørdam gave a detailed overview of the structure and classification theory for separable simple nuclear C*-algebras. Asger Törnquist then spoke on Borel parameterizations of the class of separable C*-algebras. On Wednesday morning, Todor Tsankov gave an in-depth presentation of Borel reducibility and turbulence, while Wilhelm Winter expounded the theory of nuclear dimension, which has become an important ingredient in the classification program for separable nuclear C*-algebra.

Edward Effros, one of the key figures in the initial period of contact between C*-algebras and descriptive set theory in the 1960s, offered his perspective on classification and Borel complexity at the beginning of Thursday morning's session. He spoke in particular about

quantization in functional analysis and asked what the theory of Borel complexity might be able to say about the classification of operator spaces. N. Christopher Phillips then gave an exposition of the proof of Kirchberg's $\mathcal{O}_2$ embedding theorem, which asserts that every separable exact C*-algebra embeds into the Cuntz algebra $\mathcal{O}_2$. This talk was prepared in response to a question of Törnquist, who in his Tuesday lecture had asked whether Kirchberg's theorem admits a proof consisting of procedures that are all demonstrably Borel. The interest in this issue, on which no definitive conclusion was reached, is that Farah, Toms, and Törnquist needed to devise an auxiliary trick for avoiding it in the proof of their recent result that the classification of separable exact C*-algebras lies below a group action in the hierarchy of Borel complexity.

Friday morning was largely devoted to a talk of Christian Rosendal, who discussed the problem of classifying Banach spaces up to isomorphism and the role that descriptive set theory has played in advancing the rough classification program of Gowers. Although the methods involved are unlikely to impact the study of C*-algebras at a technical level, the lecture revealed an intriguing set of analogies between the two disciplines. For example, Ferenczi and Rosenthal's dichotomy between minimality and tightness can be seen as a Banach space version of the distinction between pure infiniteness and stable finiteness in C*-algebra theory.

The first two afternoons of the workshop consisted mainly of small group sessions in which experts provided details and answered questions on various topics. These tutorials were organized on the basis of various questions which participants submitted in a session at the beginning of Monday afternoon. Edward Effros and David Sherman led a discussion on noncommutative boundary theory. Ilijas Farah gave more details on the Akemann–Weaver construction. Christian Rosendal talked about descriptive set theory and Borel reducibility. Dima Sinapova presented the method of forcing. Aaron Tikuisis and Andrew Toms talked about the Cuntz semigroup. Mikael Rørdam spoke about K-theoretic invariants and the central role of the Cuntz algebra $\mathcal{O}_2$.

Open problems were gathered in a session at the beginning of Tuesday afternoon. In the last three afternoons, the participants worked on a number of these problems in small groups. The projects included the following.

Borel complexity of the class of separable C*-algebras. This group considered the question of complexity of isomorphism relation on the following classes of separable C*-algebras: nuclear, exact, simple exact, all, all simple. It was proved by Farah–Toms–Törnquist that isomorphism of separable nuclear C*-algebras is not classifiable by countable structures, but is reducible to an orbit equivalence relation of a Polish group action (the group being $(\mathcal{O}_2)$). One of the main themes was an attempt to prove that the isomorphism of separable C*-algebras is a complete analytic equivalence relation, using the characterization of the latter by Ferenczi–Louveau–Rosendal.

Turbulence and automorphism groups of C*-algebras. This group worked on the question of the complexity of the conjugation relation (by inner or arbitrary automorphisms) on automorphism groups of C*-algebras. More precisely, the question was when is the action of (A) on itself turbulent, and how is this related to properties like $\mathcal{Z}$-stability? This group made some progress on the first part of the question, and concluded that a more complete understanding is likely to hinge on the Rokhlin property and its consequences for the structure of certain C*-algebras.

Naimark's problem. This group approached the problem from both sides: trying to give a negative answer from assumptions weaker than $\diamondsuit$, and trying to construct a model of ZFC in which there are no counterexamples. We have obtained some new information about the structure of a counterexample and explored some possible alleys for constructing a model of ZFC in which there are no counterexamples.

The generator problem. In 1967 Kadison asked whether all separable simple C*-algebras are singly generated. This group considered the question whether a generic separable simple C*-algebra is singly generated. They proved that unital, separable, simple AH algebras C*-algebras with diagonal maps are singly generated. This class contains C*-algebras that lack certain regularity properties (they need not have slow dimension growth, finite nuclear dimension, or strict comparison, and need not be $\mathcal{Z}$-stable) and includes Toms' example of a nonclassifiable C*-algebra. This shows that single generation is more general than these regularity properties, a fact that was not clear at the beginning of the workshop.

Turbulence and masas in $\mathbf{II_1}$ factors. This group considered the following question. Fix a II_1 factor M and consider the space of all of its maximal abelian subalgebras (masas). This is a Polish space in the Effros–Maréchal topology. Now consider conjugacy of masas by automorphisms of M. Is this equivalence relation turbulent? Kerr and White had given a positive answer under some additional assumptions on M. The most interesting open case is when M is the hyperfinite factor R.

Conclusion This workshop gathered researchers in descriptive set theory and classification theory of C*-algebras together. Since these two groups of researchers were almost disjoint, this gave a very exciting opportunity to start new collaborations. It was particularly fortunate that this encounter happened in AIM's informal setting. Some of the groups formed at the workshop, including some informal ones not listed above, are continuing their collaboration.