Research experiences for undergraduate faculty

organized by
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Workshop Summary

This was the fifth in a series of workshops for undergraduate faculty, and was held at the Institute for Computational and Experimental Research in Mathematics (ICERM). The goals of REUF are:

- To provide faculty participants the opportunity to have a research experience investigating open questions in the mathematical sciences, and enhance a love of doing original mathematics.
- To prepare participants to engage in research with undergraduate students at their home institutions, in some cases using the problems presented.
- To involve some participants in long-term research collaborations.
- To establish a network of faculty at primarily undergraduate institutions together with faculty at research universities who support collaboration and undergraduate research.

The four mathematical leaders, Ruth Haas of Smith College, Loek Helminck of North Carolina State University, Richard Rebarber of University of Nebraska-Lincoln, and Gizem Karaali of Pomona College, presented problems Monday morning and research groups formed Monday afternoon. The groups worked throughout the week and presented reports Friday afternoon. There were also whole group discussions about undergraduate research, the computer mathematics system *Sage*, and workshop continuation activities. Brief descriptions of the problems, excerpted from reports written by the participants, are given in Sections 1 – 4 below.

Classifying coloring graphs

Julie Beier, Janet Fierson, Ruth Haas, Carl Lienert, Heather Russell, Kara Shavo

Let G be a graph, and let $V(G)$ be the vertex set of G. A *proper k-coloring* of G is a function $f\colon V(G) \to \{1, \ldots, k\}$ such that $f(v_1) \neq f(v_2)$ whenever v_1 and v_2 share an edge in G. If there exists a proper k-coloring of G, we say that G is *k-colorable.* The *chromatic number* of G, denoted $\chi(G)$, is the minimum k such that G is k-colorable.

The *coloring graph* was developed as a way to understand certain relationships between $k-$colorings. Given a graph G and number k, the *k-coloring graph of G*, denoted $\mathscr{C}_k(G)$, is the graph with vertex set the proper k-colorings of G and edges between colorings that differ at exactly one vertex. Figure 1 has an example. The coloring graph displayed is $\mathscr{C}_2(G)$, where G is the disconnected graph consisting of a path of length 2 and an isolated vertex.

Our goal in this project is to study the following broad question: *Which graphs are coloring graphs?*

During the workshop week, we formulated many more specific questions and developed several useful techniques and thought processes in an attempt to answer them. Although all

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FIGURE 1. The 2-coloring graph for G

of these questions have not yet been fully answered, and all of these techniques not yet fully exploited, we obtained many results both during the workshop and in the days immediately following.

On the Structure of Generalized Symmetric Spaces of SL_2 (F_q)

Catherine Buell, Aloysius Helminck, Vicky Klima, Jennifer Schaefer, Carmen Wright, Ellen Ziliak

Let G be a group and θ be an order-two automorphism (an involution). We can determine the fixed points, $H = G^\theta$ and find the generalized symmetric space $G/H \cong Q = \{g\theta(g)^{-1} \mid g \in G\}$. We can then examine the orbits of various subgroups of G on Q. Also, we can determine an extended symmetric space R and discuss relations between R, Q, H, and G. For our project, we investigated $G = \mathrm{SL}_2(\mathbb{F}_q)$, the group of 2×2 matrices with determinant 1 with entries in the finite field $\mathbb{F}_q$. [Results were obtained and work continues on open questions. A paper is in preparation.]

Population Dynamics

Daniel Birmajer, Jason Callahan, Richard Rebarber, Eva Strawbridge, Cristina Villalobos, Shenglan Yuan

Consider an age structured population represented by the n-length vector, $\mathbf{x}$, where x_i is the population density of the ith population class. The population classes can represent growth stages. For example if $\mathbf{x}^T = (x_1, x_2, x_3, x_4, x_5)$, then x_i would represent infants, juveniles of three different classes, and adults for $i \in \{1, 2, 3, 4, 5\}$ respectively. Then individuals progress from a junior to senior age class with a given rate and probability of survival s_i for $i \in \{1, 2, 3, ..., n-1\}$ where, for example, s_1 represents the growth and survival of infants to juveniles, etc. This survival can be represented by the matrix A given by

$$A = \begin{pmatrix} 0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ s_1 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & s_2 & 0 & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 & s_{n-1} & 0 \end{pmatrix}. \tag{1}$$

Because A represents a survival and growth matrix, all the elements of A are non-negative and it is physically reasonable to expect that $\lim_{n\to\infty} A^n \mathbf{x}_0 = 0$. Meaning, given an initial population $\mathbf{x}_0$ in the absence of reproduction we can evolve the population in time by the relation $\mathbf{x}_{n+1} = A\mathbf{x}_n$. Without the recruitment of new individuals, we would expect the population to grow and age, and eventually all die out. This means that the spectral radius of A, $\rho(A)$, should be strictly less than one. That is

$$\rho(A) = \max\{|\lambda|, \text{ where } \lambda \text{ is an eigenvalue of } A\} < 1.$$

The problem was specialized into two models. [In addition to theoretical work,] we carried out numerical simulations for these models. We observed stability of the equilibria in the "safe zone" for both models. Further investigation is needed. There are some research directions that could be good for involving undergraduate students. Students can work on numerical experiments and/or theoretical explanations at their own level.

Math for Social Justice: Saving Life and Love through Engagement

Nicholas Boros, Rhonda Ellis, Gizem Karaali, Karen McCready, and William Miles

Problem: How does community engagement correlate with crime rate and divorce rate?

What do we mean by community engagement? Indicator variables:

- religious organizations or places of worship
- informal gathering spaces (beauty salons, cafes, parks, corners)
- sports or recreational clubs or teams specific to an ethnic or cultural community
- social or cultural clubs
- restaurants, grocery stores, or specialty stores that serve or sell products specific to an ethnic or cultural community
- arts and cultural venues or public art that celebrate a specific ethnic or cultural heritage
- nonprofit organizations that serve a specific ethnic or cultural community
- festivals or parades that express or celebrate the heritage or the presence of an ethnic or cultural community
- places where people can purchase books or music relating to an ethnic or cultural community
- major institutions or parks that celebrate the cultural heritage of an ethnic or cultural community
- educational institutions

Why did we choose to focus on community engagement?

There was a study in Malcolm Gladwell's book *The Outliers* that followed a small community in Rozeto, Italy, where people had a significantly longer lifespan in comparison to most other communities, even after moving to Pennsylvania. It was the author's conjecture that perhaps this could be attributed to having strong relationships with each other in the community, after ruling out most other factors. Furthermore, most of these community engagement factors can be easily replicated in most communities without requiring much funding. So if they do in fact contribute to better divorce rate, crime rate, life span, etc., then this would good news. To simplify things we decided to focus on crime rates and divorce rates and build an index function to measure community engagement.

Where do we get data? Data can be siphoned relatively easily from that collected by the US Census Bureau. This is accomplished using an application called "Data Ferrett". Assuming that our scores correlate well with crime rate and divorce rates it may be possible to then optimize the weights to correlate even better to those two particular response variables.

Once we get an approach that seems to work, it would not be too hard to involve undergraduates in the process. This is because it is quite easy to get good data from the US Census Bureau and the background mathematics/statistics needed up to this point are quite elementary. However, we still are continuing to seek novel approaches to the problem.