Recursion structures in topological string theory and enumerative geometry

organized by
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Workshop Summary

Summary. The goal of this workshop is to solve the mystery that is exciting a number of researchers working on topological string theory, integrable systems, and enumerative geometry. The central theme is a newly discovered *topological recursion* of Eynard and Orantin [EO1] that was originated in Random Matrix Theory [E1], and its geometric and string-theoretic realization due to Bouchard, Klamm, Mariño, and Pasquitti [BKMP,BM, Mar]. The workshop was aimed at solving some of the concrete conjectures in Gromov-Witten theory and enumerative algebraic geometry proposed in these papers. The workshop was filled with the energy of the young participants. The enthusiasm was apparent from the beginning, and continued throughout. As a result of heated discussions and extended hours of explanations, the workshop was able to achieve a large amount of planned goals.

Achievements. Significant achievements of the workshop include the following.

(1) As a result of the collaborations promoted by the workshop, one of the proposed conjectures (the Bouchard-Mariño conjecture on Hurwitz numbers [BM]) has been already solved [BEMS, EMS, MZ].

(2) During the workshop ample time was allocated for the participants to understand the new idea of topological recursion from its originator, Bertrand Eynard, and his collaborators Nicolas Orantin and Gaëtan Borot. Since the idea is new to most of the mathematicians and physicists at the workshop, all the introductory talks and extended explanations were enthusiastically received.

(3) The relation between the topological recursion of [BKMP, BM, EO1] and the *Virasoro conjecture* is clarified. Note that the Virasoro constraint condition does not carry any information for the Gromov-Witten invariants of Calabi-Yau 3-folds. If it doesn't contain any information, then why do we care? The emerging consensus at the workshop is that, at least for the case of *toric* Calabi-Yau 3-folds and their degenerations, the topological recursion *is* the equation that recursively computes the Gromov-Witten invariants, and that it occupies the important position, replacing the Virasoro constraint condition.

(4) The puzzling notion of *spectral curves* of the topological recursion [E1, EO1] has been cleared to the integrable system community. Since the same terminology already exists in the theory of integrable systems, many researchers have been confused with the notion. A more unified understanding is being developed, and the role of the KP equations in topological recursion is identified. A solution to the topological recursion is expected to satisfy the modular property and the holomorphic property simultaneously, solving the BCOV holomorphic anomaly equation [DV, Mar]. It is the current understanding that a solution to the topological recursion also satisfies

the KP equations, and hence it admits an expression in terms of Riemann theta functions. The modularity and holomorphicity follows from this expression.

(5) The significance of Laplace transform in the topological recursion is one of the key discussions at the workshop. The Laplace transform of the Mirzakhani recursion formula for the Weil-Petersson volume of the moduli spaces of bordered hyperbolic surfaces [Mir] is a topological recursion on the sine curve as its spectral curve [E2, EO2]. It is reported at the workshop that in a rather similar way the Laplace transform of a combinatorial formula known as the *cut-and-join equation* is a topological recursion on the Lambert curve [EMS, MZ]. After taking the Galois average, this topological recursion reduces to the Bouchard-Mariño conjecture [EMS].

(6) The participants are reminded of the power of Random Matrix Theory, or matrix models. The idea of [BKMP, BM, EO1] is to use the topological recursion *as an axiom*, without having any matrix integral in mind, and obtain geometric invariants. Yet for the case of Hurwitz numbers, indeed *there is* a matrix model discovered in [BEMS] that produces the topological recursion of Bouchard-Mariño [BM]. Eynard and Orantin have also announced that there is a matrix model for a general case of the BKMP conjecture [BKMP].

Plan. We can see how fast the subject has been developing by comparing the achievements and our original plan. The following is an excerpt from our proposal:

"Consider the A-model topological string on a toric Calabi-Yau threefold, in which case the mirror B-model geometry is controlled by a family of algebraic curves. It was then conjectured [BKMP,Mar], from physical principles, that applying the recursion to this family of curves should generate the open and closed amplitudes of the B-model; the conjecture was verified computationally in various examples. Through mirror symmetry, these amplitudes become generating functions of open and closed Gromov-Witten invariants of the mirror toric threefold. Hence, this line of idea implies that, somewhat miraculously, the recursion should govern open and closed Gromov-Witten theory on toric Calabi-Yau threefolds! These exciting developments rely on arguments borrowed from physics [BKMP,DV,Mar], and cry for a mathematical explanation.

"The recursion also plays an intriguing role in Hurwitz theory [BM]. By considering a particular limit of topological string theory, it was conjectured that the recursion also computes generating functions of simple Hurwitz numbers, when applied to a particular analytic curve — the curve which implicitly defines the Lambert W-function. While it is well known that generating functions of simple Hurwitz numbers are determined by the cut-and-join differential equations, the recursion may be regarded as providing a new and unexpected explicit solution to these equations. However, the relation between the cut-and-join equations and the recursion seems to be rather intricate and involve highly non-trivial combinatorics. The manifestation of the recursion in Hurwitz theory remains mysterious; connections with integrable systems may be at the heart of the matter.

"It is the aim of this workshop to lay down the mathematical foundations behind the appearance of the recursion in enumerative geometry and topological string theory, which seem to involve new exciting interactions between geometry and integrable systems."

Possible future directions. Since the speed of developments is so fast, we are not confident in proposing a meaningful list of concrete problems. A few questions that come to our mind are the following:

(1) Noting that a number of topological recursions are obtained as the Laplace transform of something interesting, it is natural to ask *what is the inverse Laplace transform of a more general topological recursion?* The idea of [EMS, MZ] is that the spectral curve is the *Riemann surface* of the natural holomorphic function that is obtained by the Laplace transform. Thus the mirror curve of a toric Calabi-Yau 3-fold should also be the Laplace transform of something. In this sense the Laplace transform is appearing as a mirror map here.
(2) The theory of open Gromov-Witten invariants is currently growing to be a gigantic theory, mainly due to the analysis difficulties. Since the topological recursion provides some algebraic structures to these invariants, does it provide also a better understanding?
(3) The topological recursions so far discussed are all related to multi-component KP equations. Are there any generalization of the topological recursion corresponding to other integrable systems, such as those suggested by the program of Yongbin Ruan on the geometry of integrable systems? How can we characterize the solutions of topological recursions as τ-functions of multi-component KP equations in terms of the Sato Grassmannians and/or other structures?

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