# Rational Catalan combinatorics

organized by
Drew Armstrong, Stephen Griffeth, Victor Reiner, and Monica Vazirani

Workshop Summary

## 1. Topics of the Workshop

The mathematics associated with the classical Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$ is broad and surprisingly deep. In recent years this mathematics has become systematized under the framework of reflection groups.

**1.1. RCA** At the most general level, let $G$ be a finite group acting irreducibly on $V = \mathbb{C}^n$ and generated by reflections (where a **reflection** is a unitary operator fixing a hyperplane). Let $N$ be the number of reflecting hyperplanes and let $N^*$ be the number of reflections in $G$. We define the **Coxeter number** of $G$ by

$$h = \frac{N + N^*}{n}.$$

Recall that ring of coinvariants $\mathbb{C}[V]/\mathbb{C}[V]^G_+$ is a graded version of the regular representation of $G$. Hence if $U$ is any irreducible representation of $G$, then $U$ appears with multiplicity $\dim U$ in the coinvariant ring. Let

$$e_1(U) \leq e_2(U) \leq \cdots \leq e_{\dim U}(U)$$

be the degrees in which $U$ occurs, and call these the **exponents** of $U$. If $e_1 \leq e_2 \leq \cdots \leq e_n$ are the exponents of the defining representation $V$ (called the exponents of $G$ itself), then we call $d_i := e_i + 1$ the **degrees** of $G$.

The Weyl algebra $\mathbb{C}[V \oplus V^*] \rtimes G$ encodes the action of $G$ on differential operators. The Cherednik algebra $H_c(G)$ is the universal deformation of this algebra. For general parameters $c \in \mathbb{C}$, $H_c(G)$ has no finite-dimensional representations. However, Berest, Etingof and Ginzburg proved that for a real reflection group $G$ and each positive integer $p$ equal to 1 modulo to the Coxeter number $h$, there exists a special finite dimensional representation of $H_{p/h}(G)$ called $L_{p/h}(\mathsf{triv})$, and calculated its $W$-equivariant Hilbert series. Griffeth and Gordon later proved analogous results for all reflection groups $G$ and positive integers $p$ coprime to $h$. In this generality, the Hilbert series of $L_{p/c}(\mathsf{triv})^G$ equals

$$\mathrm{Cat}_q(G, p) := \prod_{i=1}^{n} \frac{[p + e_i(V')]_q}{[d_i]_q}, \qquad (1)$$

where $V'$ is an irreducible representation of $G$ of dimension $n$ (another reflection representation, closely related to the defining representation $V$) and $[A]_q = 1 + q + q^2 + \cdots + q^{A-1}$ is the standard $q$-integer. We call (1) the **rational Catalan number**. Various simplifications occur in special cases. For example, if $G = \mathfrak{S}_a$ is the symmetric group with Coxeter number

$h = a$ and if $p = b$ coprime to $a$, then we have

$$\mathrm{Cat}_q(\mathfrak{S}_a, b) = \frac{1}{[a+b]_q} \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q = \frac{[a+b-1]_q!}{[a]_q!\,[b]_q!}, \tag{2}$$

where $[A]_q! = [A]_q[A-1]_q \cdots [2]_q[1]_q$ is the standard $q$-factorial. (The symmetry in (2) between $a$ and $b$ is interesting and *not obvious*.) Finally, as a special case of this we recover the classical $q$-Catalan number

$$\mathrm{Cat}_q(\mathfrak{S}_n, n+1) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q. \tag{3}$$

**1.2. Enumeration** At the other end of the spectrum, Nicolaus Fuss showed in 1791 that the numbers

$$\mathrm{Cat}(\mathfrak{S}_n, mn+1) = \frac{1}{n}\binom{(m+1)n}{n-1}$$

enumerate the dissections of a convex $(mn+1)$-gon into $(m+2)$-gons. (The case $m = 1$ of polygon triangulations was studied earlier by his grandfather-in-law, Leonhard Euler.) Since then the Catalan numbers and their relatives have played a central role in enumerative combinatorics. Richard Stanley maintains a catalogue with hundreds of examples of interesting structures counted by Catalan numbers.

In recent years, several families of such objects have begun to unify under the language of root systems and reflection groups. To name three examples:

(1) The classical **lattice of noncrossing partitions** has been generalized to reflection groups. The structure of this poset is related to Garside structures, which provide classifying spaces and normal forms for Artin braid groups. Chain enumeration in the poset is related to Hurwitz type problems and the Lyashko-Looijenga map on the discriminant hypersurface.
(2) The classical **nonnesting partitions** are defined as antichains in the poset of positive roots of a Weyl group. They are related to combinatorial labelings for the regions of the Shi hyperplane arrangement and the theory of parking functions. This is the combinatorics most clearly underlying the representation theory of RCA's. However, it seems not to generalize to non-crystallographic groups.
(3) The classicial **associahedron** is a polytope whose incidence relations describe the dissections of a convex polygon. It was introduced by Stasheff as a basis for the homology of H-spaces. Fomin and Zelevinsky have generalized the associahedron to all root systems via their theory of cluster algebras.

The occurrence of the Catalan numbers in example (2) is well-understood. However, Catalan formulas that occur in examples (1) and (3) have only been proved case-by-case using the classification of reflection groups. This is deeply unsatisfactory. A workshop was held at AIM in January 2005 ("Braid Groups, Clusters and Free Probability") to examine these numerological coincidences. Since 2005, the subject of **Catalan combinatorics** has grown up to try to unify the examples (1), (2), (3).

**1.3. Diagonal Harmonics**. At the intersection of RCA theory and Catalan combinatorics, there is an extremely rich combinatorial theory that so far is specific to type $A$ (the symmetric group). This is the theory of **diagonal harmonics** and the $q,t$-**Catalan numbers** of Garsia and Haiman. The central objects here are the nabla operator and the Macdonald symmetric functions. Recently there has emerged a $q, t$-version of the type $A$ rational Catalan number: $\mathrm{Cat}_{q,t}(a, b)$. Both of the symmetries

$$\mathrm{Cat}_{q,t}(a, b) = \mathrm{Cat}_{t,q}(a, b)$$

and

$$\mathrm{Cat}_{q,t}(a, b) = \mathrm{Cat}_{q,t}(b, a)$$

are interesting, non-trivial, and worthy of further study.

## 2. Goals of the workshop

The program of "Catalan Combinatorics" emerged from the January 2005 AIM Workshop on "Braid Groups, Clusters and Free Probability". In short: The same Catalan numerology had emerged in several different subjects and everyone wanted to know why. Since 2005 there have been several important breakthroughs, and the Catalan Combinatorics program has grown closer to the subject of $q, t$-Catalan numbers and diagonal harmonics. However, a unifying framework is still missing.

The three cultures working on the theories of **rational Cherednik algebras**, **Catalan combinatorics** and **diagonal harmonics** have until now been mostly disjoint. The main goal of the Dec 2012 workshop on "Rational Catalan Combinatorics" was to bring these three cultures together, in order to

- search for a unifying framework for Catalan combinatorics,
- stimulate each subject with outside points of view,
- catalyze progress on open problems,
- discover common interests and foundations for future collaboration, and
- open future possibilities in all three subjects.

The strength of the RCA approach is that one can define rational Cherednik algebras for all complex reflection groups, and the representation-theoretic phenomena they share are case-fee (i.e., independent of the Shephard-Todd classification). We hoped that the flexibility and generality of rational Cherednik algebras would suggest a unification of Catalan phenomena, under a new framework that we termed **Rational Catalan Combinatorics**. The theory of diagonal harmonics, being closely related to both RCA and Catalan combinatorics, may serve as a bridge between the two points of view.

## 3. Outcomes of the Workshop

**3.1. Workshop activities** There were two lectures each morning except for Friday, which had a single lecture. In addition to the lectures, there were guided exercise sessions on the first two afternoons. Since the workshop participants were coming from somewhat disjoint backgrounds, we felt it was important to get some hands-on experience with the objects of the workshop. Several participants remarked that these exercise sessions were helpful. The problems and solutions from the sessions will be available on the workshop website.

On the Wednesday, Thursday and Friday afternoons we broke into small working groups. We came together twice (on Wednesday and Friday afternoon) for moderated problem sessions, to record and discuss the main open problems and questions. These problems will also be available on the workshop website.

The following was the schedule of the workshop:

| | Morning | Afternoon |
|---|---|---|
| Monday | Talks by Vic Reiner and Iain Gordon | Parallel exercise sessions run by Gwyn Bellamy and Christian Stump |
| Tuesday | Talks by Drew Armstrong and Mark Haiman | Exercise session run by Brendon Rhoades and Nathan Williams, followed by moderated problem session |
| Wednesday | Talks by Francois Bergeron and Eugene Gorsky | Working groups |
| Thursday | Talks by Iain Gordon and Adriano Garsia | Working groups |
| Friday | Talk by Roman Bezrukavnikov | Moderated problem session, followed by working groups |

**3.2. Working groups** Before breaking into working groups on Wednesday, Thursday and Friday afternoon, a moderator wrote down the proposed working topics and then participants voted on which working group to join. In the first round everyone voted as many times as they wished. In the second round the topics were read in increasing order of votes received, and each participant could only vote once. The AIM staff then sent the working groups to appropriate areas based on their size.

Here are the topics discussed during the working groups:

- Posets of rational noncrossing partitions
- $q, t$-combinatorics of parallelogram polyominoes
- A nabla operator for wreath Macdonald polynomials
- Degenerating a distinguished basis from a finite-dimensional DAHA representation to the RCA
- Open problems for type A rational $q, t$-Catalan numbers $\mathrm{Cat}_{q,t}(a, b)$, including the invertibility of the Zeta map
- Lapointe's type B Macdonald polynomials
- Main Conjecture from the paper "Parking Spaces" and the magic HSOP
- Rational Tamari posets
- $\mathrm{Cat}(a, b)$ structures when $\gcd(a, b) = d > 1$

**3.3. Collaborations** Several working groups reported progress on their problems and the beginnings of new collaborations. Haiman had some interesting discussions with Bezrukavnikov that led him to formulating a new plan of attack on a current project. Assaf, Bergeron and Niese are working a paper that arose from their Tamari combinatorics group. Various subsets of Armstrong, Fishel, Gorsky, Mazin, Vazirani, Warrington and Williams are working on papers about the Zeta map on parking functions and the Pak-Stanley bijection to the Shi arrangement, and extending these results from $(a, ma \pm 1)$ to $(a, b)$ coprime.

More collaborations will become clear when the workshop website is up and running.

**3.4. Future Impact** This workshop discussed problems that are of deep interest from at least three different points of view. Groups of researchers came together who normally would not meet (Bezrukavnikov and Pak, for example, have known each other since high school, but this is the first conference that they have both attended). We learned about exciting work going on in closely related areas (Eugene Gorsky's talk, in particular, inspired many people). It too early yet to see, but we expect that the collaborations born at this workshop will have an important impact on the future of the field.