Random matrices

organized by
Terence Tao and Van Vu

Workshop Summary

This workshop, sponsored by AIM and NSF, focused on recent developments on limiting distributions concerning the spectrum of a random matrix. We devoted most of the time on the following two types of limiting distributions:

(A) Global: One would like to understand the limiting law of the counting measure generated by all eigenvalues. The most famous example here is the semi-circle law regarding the eigenvalues of random Hermitian matrices, discovered by Wigner in the 1950's.

(B) Local: One would like to understand the limiting law of fluctuation of individual eigenvalues (say the largest or smallest eigenvalues, or in general, the k^{th} eigenvalues for any k), or local interaction among eigenvalues in a small neighborhood. Typical examples here are the Tracy-Widom law (for the extremal eigenvalues) and Dyson laws (for the distribution of gaps between consecutive eigenvalues and for correlation functions).

In the workshop, we began with an overview about recent developments that have established (both global and local) universality in many important cases. Next, we discussed the techniques used in the proofs. Another important activity was to set new targets and discuss open problems. We were mostly concerned with models of Hermitian matrices where the upper-diagonal entries are iid random variables with mean 0 and variance 1, although in a few talks, different or more general models have been considered.

The main theme of the workshop is "Universality". Universality is a general phenomenon which asserts that limiting distributions of eigenvalues do not depend strongly on the model of matrices. For instance, the semi-circle law holds for all random symmetric matrices with iid entries having mean zero and variance one. It is thus very easy to use this phenomenon to make conjectures. One only needs to compute the desire distribution for one model (typically GUE , GOE or GSE where an explicit formula for the joint distribution of the eigenvalues is available thanks to Ginibre), and then conjecture that the same distribution holds for others (under some mild regularity hypotheses, of course). Most conjectures of this type have been open for several decades until very recently, when works of Erős et. al. and Tao and Vu have settled several of them.

The workshop started with a survey talk by P. Deift, who talked about universality of the invariant ensembles. This talk discussed the invariant models GUE, GOE and GSE and provided several useful insights and identities. The next talk was by Vu, who talked about the universality of random matrices with iid entries. Vu's talk surveyed many recent results, and focused on the Four Moment Theorem by Tao and Vu. This theorem, proved in 2009, asserts that all local statistics of eigenvalues depend only on the first four moments of the entries. As applications, one can prove, among others, the universality of Gaussian fluctuation of eigenvalues, the universality of Tracy-Widom law at the edge of the spectrum and that of Dyson law in the bulk. The proof of the Four Moment theorem was later sketched by Tao.

Another set of talks on Universality was delivered by Erdős, Schlein and Yau on their work on the universality of Dyson law, using methods inspired by statistical physics, in particular using log-Sobolev methods via the method of local relaxation flows to show rapid relaxation to local equilibrium of the Dyson Brownian motion. This is a very general technique that has established universality for a wide class of Wigner-type ensembles.

Several other talks were given by various researchers. Wood on the universality of the Circular Law, Dumutriu and Tran on the Universality of the semi-circle law of random regular graphs, Soshnikov on the moment method and universality at the edge also on universality results for product of random matrices. Ben Arous spoke on a surprising application of GUE statistics to random Morse functions and spin glass models, while Silverstein showed how the universal laws for random covariance matrices could be used for statistical estimation.

The afternoon sections have been filled with numerous discussions and open problem sections, some of which were led by junior participants. In particular, Wang, gave a interesting talk about a model of random matrices arose from computational biology. The open problems focused on the a variety of topics, including the following:

- **Eigenvectors.** While we now know a considerable amount about eigenvalues, little has been proved for the eigenvectors. The case of GUE and GOE is well understood, thanks to earlier works of Jiang and others. However, almost nothing is known for the general case, until very recently when a few partial results have been obtained (Erdős et. al., Tao-Vu, Vu-Wang). These results, in some sense, showed that a typical eigenvector is close a random unit vector. However, there are still lots of problems remaining. Another deep, and important, problem is to study the delocalization of eigenvectors of a random Schrödinger operator; Erdős surveyed the state of knowledge in this area (which is analogous to, but far more difficult than, the delocalization of eigenvectors of Wigner matrices, which played a decisive role in many of the recent advances in this subject).
- **Random matrices under perturbation.** The main question here is: How much does the spectrum of a random matrix change under a small perturbation? (For instance, one can add a low rank or very spare matrix.) An interesting question in this direction (existence of outliers) was solved in the workshop by Tao, using a suggestion from Guionnet and a formula that appeared in Deift's talk.
- **Spectral properties of random matrices with dependent entries.** The general problem here is to investigate random matrices whose entries are not entirely independent. One model is the adjacency matrix of a random regular graph. Another model is random block matrices, where some of the blocks may be repeated, or vanish entirely. We know little about these models. For instance, the semi-circle law for the sparse random regular graph was proved only few months before the workshop (and which was presented during the workshop). At this moment, we have not yet obtained any information about the local statistics.
- **Heuristic models of eigenvalues.** The eigenvalues of GUE are known to be individually distributed in an asymptotically normal fashion, thanks to the work of Gustavsson, but are not independent; there are both short-range and long-range correlations between these eigenvalues. During the workshop, a non-rigorous model for these eigenvalues (based on Taylor expansion of the Hamiltonian) was developed

which provided a heuristic explanation for many observed phenomena for these eigenvalues, such as the distribution of the largest and smallest eigenvalue gaps.