Gaps between primes

organized by
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Workshop Summary

Theme of the Conference

The conference was centered around a recent results of Dan Goldston, János Pintz, and Cem Yıldırım on small gaps between primes. To describe these results, let p_n denote the n^{th} prime number. The three major results of Goldston, Pintz, and Yıldırım that were discussed were the following. The first is that

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = 0. \tag{1}$$

The second result is a quantitative strengthening of the first; i.e.,

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{(\log p_n)^{1/2}(\log\log p_n)^2} < \infty. \tag{2}$$

The third result is a conditional strengthening of (1). Assume that the primes have a level of distribution $\theta > 1/2$. Then there is some constant $C = C(\theta)$ such that

$$\liminf_{n\to\infty} p_{n+1}-p_n \le C(\theta). \tag{3}$$

In particular, one may take $C(1-\epsilon) = 16$ for ϵ sufficienlty small.

Descriptions of Talks

(1) **Kannan Soundarajaran** (Monday 9 am and 2 pm) begin the conference by giving a theoretical framework for short gaps between primes. He discussed Cramer's probabilistic model for the distribution of primes, pointed out its shortcomings, and discussed how they could be addressed. He continued his remarks in a second talk on Monday afternoon. These talks constituted an expanded version of a talk that he gave later (January 2005) at an AMS meeting in San Antonio. These remarks were published in his article "Small gaps between prime numbers: The work of Goldston-Pintz-Yıldırım, Bulletin of the American Mathematical Society, January 2007, 1–18.

(2) **Hugh Montgomery** (Monday 11 am) provided historical context for the Goldston-Pintz-Yıldırım result by giving a survey on known results about gaps between primes.

(3) **Dan Goldston and Andrew Granville** (Monday 4 pm and Tuesday 9 am) gave a full account of a conditional result (3). Another way of stating this result is that it shows that if a k-tuple $\{h_1, h_2, h_3, \ldots, h_k\}$ satisfy a certain obviously necessary condition ("admissibiity"), then there are infinitely many integers n such that at least one of $n+h_1, n+h_2, \ldots, n+h_k$ is prime. Although this result is conditional, it illustrates the main ideas behind the collection of the other results (1) and (2).

(4) **Kannan Soundarajaran** (Tuesday 11 am)gave an tutorial on the Selberg upper bound sieve, which is a key components the proofs of (1), (2), and (3). He also

showed that the level of distribution $\theta = 1/2$ is a true barrier for obtaining primes in tuples, at least for the current argument. Of course, it is possible that one could devise more efficient arguments.

(5) **Robert Vaughan** (Tuesday 2 pm) gave a tutorial on the Bombieri-Vinogradov Theorem, which is another key component in the GPY proof. This theorem says that the sequence of primes have a level of distribution 1/2. The version he presented was an expanded version of his 1979 Acta Arithmetica paper.

(6) **Cem Yıldırım** (Wednesday 9 am) presented an account some of the technical issues that arise in the GPY proof; iin particular, he discussed how certain complex integrals with multiple variabels arise in the course of the argument, and he showed how they could be evaluated.

(7) **János Pintz** (Wednesday 11 am) gave an exposition on the result in (2). In the course of his lecture, he also showed how one could prove the weaker result (1) without making an appeal to Gallagher's result on averages of the singular series. Such an argument is essential to the proof of (2), for no suitable analog of Gallagher's result is available for (2).

(8) **John Friedlander** (Wednesday 2:30 pm) lectured on the Bombieri-Friedlander-Iwaniec results on level of distribution for general sequences. Their results have some resemblance to the conditional results that imply bounded gaps between primes. Friedlander gave a frank discussion of the significant technical obstacles in the way of completing a proof along these lines.

(9) **Andrew Granville and Kannan Soundarajaran** (Thursday 9 am) discussed combinatorial sieves. These present an alternative to the Selberg upper bound sieve, and could conceivably be incorporated into improvements of (2) or into unconditional versions of (3).

(10) **Yoichi Motohashi** (Thursday 2 pm) presented a version of the Selberg sieve with a bilinear form of the remainder term. This form of the sieve could possibly be an important ingredient in a uncondtional proof of (3).

(11) **Sidney Graham** (Thursday 3 pm) discussed his joint work with Goldston, Pintz, and Yıldırım on gaps between E_2-numbers, which are numbers that have exactly two prime factors. Let q_n be the n^{th} number with exactly two prime factors. They have proved that

$$\liminf_{n\to\infty} q_{n+1} - q_n \leq 6.$$

(12) **Harald Helfgott** (Friday 11:30 am) spoke on an interesting conjecture about the large sieve. The example of sifting for squares shows that the large sieve is sharp in a situation where a positive proportion of residue classes are removed for each prime. However, Helfgott argues that this situation is atypical because of the inherent underlying algebraic structure of the squares. He conjectures that for sets without this algebraic structure, the sieve bound is much smaller, and he can prove this for an analogous two-dimensional problem.

(13) **Dan Goldston** (Friday 2 pm) wrapped up the conference with an overview of the talks, an overview of the results of the working groups, and a discussion of open problems on short gaps between primes.

Reports from Working Groups

(1) **De Polignac Numbers** De Polignac conjectured that every even number is a difference of two primes in infinitely many ways. Goldston proposed the problem of proving that the set of De Polignac numbers has positive density. The working group, lead by K. Soundarajaran, was able to prove this on the assumption of the Elliott-Halberstam conjecture. A fairly simple argument shows that the density is at least 1/75 of all even numbers. With a more elaborate argument, the group was able to improve this to 8/75. The proof uses the conditional 6-tuple result of Goldston, Pintz, and Yıldırım. For E_2 numbers, the corresponding result is true unconditionally for 3-tuples. In this case, one can easily prove that at least 1/3 of all even numbers are de Polignac numbers.

(2) **Ratios of shifted primes.** P. Elliott proved that for a fixed a, there exists some k such that $a^k = (p+1)/(q+1)$ has infinitely many solutions in primes p, q. K. Ford asked if the GPY machinery could prove this result with a good bound on k. He lead a group that attempted a proof of Elliott's result with $a = 2, k < \epsilon \log p$. The group quickly identified the primary issue, which turns out to be understanding the singular series associated to linear forms $2n + 1, 4n + 1, 8n + 1, \ldots$. The group made some progress, but identified some difficult issues that have to be resolved. The problem appears difficult but not hopeless.

(3) **An analog of GPY for polynomials.** C. David led a large group, consisting mostly of postdocs and graduate students, that investigated analogs of GPY for polynomials. Instead of considering k-tuples of the form $\{p+h_1, \ldots, p+h_k\}$, the group considered k-tuples of polynomials $\{f_1(p), \ldots, f_k(p)\}$. Their objective was to work through the heuristics for evaluating the sums that come up when the GPY machinery is applied to the polynomial k-tuples. The results are highly speculative, as they require deep unproved results about the distribution of primes in polynomials and corresponding analogs of the Bombieri-Vinogradov theorem. Nonetheless, the group did succeed in getting some heuristic arguments, and they identified some interesting unexpected behavior on the associated singular series.

(4) **Strings of Consecutive Divisors.** A group led by C. Elstholtz and S. Graham looked at the issues that would need to be resolved to get a proof that $d(n) = d(n+1) = d(n+2)$ infinitely often. If one assumes the general Hardy-Littlewood conjecture for prime k-tuples, there are numerous constructions that lead to a proof of this result. The group looked at a number of these constructions, and they speculated on how they might be combined with possible results on E_2's to lead to an unconditional proof. The group also speculated on how new result on E_2's might lead to new results on the consecutive values of the Liouville function $\lambda(n)$.

(5) **Bombieri-Vinogradov averaged over residue classes.** One idea proposed for proving bounded gaps between primes was to prove a version of the Bombieri-Vingradov theorem with extra averaging over a set of residue classes. A group led by R. Vaughan looked at this problem. They were able to develop a line of attack that appeared likely to give a Bombieri-Vinogradov theorem of the desired type. However, they later realized that their proposed result would not lead to any improvements on gaps between primes.

(6) **Motohashi's idea for bounded gaps.** Motohashi proposed another idea for proving bounded gaps between primes. His idea was to examine the terms that give a negative contribution in the basic sum considered by Goldston, Pintz, and Yıldırım.

A. Granville and K. Soundarajaran made some preliminary calculations on these terms. They focused on estimating the contribution of terms with small prime divisors, and they were able to show that those will not contribute enough to successfully solve the problem. In other words, this approach will not succeed unless one can take account of terms with large prime divisors; “large” meaning on the order of $N^{1/10}$ or larger.