The geometry of the outer automorphism group of a free group

organized by
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Workshop Summary

Free groups and surfaces groups are fundamental objects of mathematics, and it is thus important to understand their outer automorphism groups. The combinatorial simplicity of the free group can in some instances make it and its outer automorphism group $\mathrm{Out}(F_k)$ somewhat easier to understand; however more often the additional geometric structure of surfaces make a surface group, together with the mapping class group (the outer automorphism group of a closed surface group), more accessible. This is strikingly illustrated by the current disparity between the development of a metric theory for the mapping class group and of that for $\mathrm{Out}(F_k)$. With the ultimate hope of narrowing this gap, this workshop assembled several experts in the mapping class group and Teichmüller theory together with specialists of $\mathrm{Out}(F_k)$ and geometric group theory.

The week began with an introduction to Culler-Vogtmann's Outer space CV_n, the $\mathrm{Out}(F_k)$ analogue of Teichmüller space, and its Lipschitz "metric," presented in a talk by Mladen Bestvina. This was immediately followed by a talk of Mark Feighn outlining some candidates for an analogue to a complex of curves, along with a description of a δ-hyperbolic graph he had discovered with Bestvina. These two introductory talks were intended to familiarize those from outside the area of $\mathrm{Out}(F_k)$ with the basic topological objects associated to $\mathrm{Out}(F_k)$, and with the immediate geometric problems they pose. The remaining morning talks of the week concerned recent research of the speakers.

Afternoons were almost entirely devoted to discussion sessions. Initial resistance to breaking into smaller groups was overcome by the end of the first day, and for the remainder of the week afternoon activity generally occurred on the level of two to four independent groups. A Tuesday afternoon session soliciting open problems from participants inspired several of these afternoon discussions. One such example concerned the problem of understanding folding lines in Outer space, as these provide examples of geodesics in CV_k equipped with the Lipschitz metric. In her thesis, Yael Algom-Kfir studied periodic folding lines which are axes of fully irreducible elements of $\mathrm{Out}(F_k)$; in particularly she showed that they are strongly contracting [un:A-K]. This result prompted a question, and resulting group discussion, concerning *aperiodic* folding lines; this in turn inspired Martin Lustig to write a short preprint on the subject [un:Lustig] on the very same day. A related topic investigated by one group involved how "thick" folding lines in Outer space should be characterized, as well as how to define the "thick part" of CV_k. These problems are all strongly motivated by well-known descriptions of Teichmüller space and its geodesics.

Another lively afternoon discussion addressed the possibility that $\mathrm{Out}(F_k)$ has uniform uniform exponential growth. Some very preliminary progress in this direction was made by Clay and Pettet [ar:CP] prior to the workshop: they gave a proof that non-trivial products of Dehn twist elements in $\mathrm{Out}(F_k)$ with corresponding "filling" Bass-Serre trees generate free

groups. However their result depends on passing to possibly high powers of the Dehn twists, and bounding these powers would be essential to establishing uniform uniform exponential growth of subgroups containing such twists.

Perhaps the most persuasive evidence that $\mathrm{Out}(F_k)$ might have uniform uniform exponential growth is that it is a property of the mapping class group. In a Tuesday morning talk Johanna Mangahas explained her proof of this result. Matt Clay followed Mangahas, explaining carefully that the need in the Clay-Pettet theorem for passing to powers of Dehn twists arises from a lack of control on the bounded backtracking of maps between certain trees associated to the twists. This inspired Kasra Rafi to ask whether these trees could be interpreted using 3-manifolds, thereby possibly removing the appearance of bounded backtracking.

Following Rafi's suggestion, one afternoon discussion group investigated how a new proof of Clay-Pettet's theorem might be developed using 3-manifolds. The session ended with the formulation of some interesting and promising ideas towards this end. A strategy using 3-manifolds carried over into some discussions later in the week, giving possible interpretations of the curve complex candidates, as well as an idea for an analogue of subsurface projection for free groups. For the latter topic, Minsky was an invaluable resource for properties that should be sought from subsurface projection.

Attention was also drawn to the question of whether the dual lamination of an indecomposable $\mathbb{R}$-tree in $\overline{CV}_k$ is minimal. One of the discussion groups made some progress, using work by Coulbois-Hilion-Lustig [ar:CHL-1,ar:CHL-2] and by Reynolds [un:R]. Indeed, the important case of Levitt type $\mathbb{R}$-trees seems to be solved.

This workshop seemed successful in igniting exchange between experts from the different areas. We do believe that some of the discussions begun here will continue and develop further into some collaborative work. The verbal feedback we received from participants was very positive, and we have even been asked a few times whether we will consider holding another workshop on this topic at AIM. Although several were skeptical at first, many participants were clearly believers in the workshop format by the week's end.

Bibliography

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