Nonnegative Matrix Theory: Generalizations and Applications

organized by
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Workshop Summary

The workshop brought together mathematicians with a common interest in nonnegativity as it arises in linear algebra, operator theory and max algebra. The goals included making progress on important open problems, identifying new directions and challenges, developing global themes that tie notions of nonnegativity together, as well as exchanging and comparing ideas in the study of nonnegativity of linear and non-linear maps.

The activities included morning presentations that focused on reviewing classical results and bringing the participants up to date on the status of open problems and related research. The afternoons were mostly devoted to group work, reporting of efforts and results to all participants, as well as informal discussions.

Workgroups

Several dynamic groups formed during the workshop that worked on specific problems. The groups reported their progress, questions and arising challenges regularly. Below we summarize some of that activity.

Group 1

This group worked on a conjecture by Miki Neumann, arising in the quest of a solution to the nonnegative inverse eigenvalue problem. Let $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ be the spectrum of a nonnegative matrix A with spectral radius $\lambda_0 = 1$. Is it true that

$$\prod_{j=1}^{n}(1-\lambda_j) \ \leq \ \sum_{j=1}^{n}(1-\lambda_j) \ \ ?$$

The group reported that the answer is in the affirmative if any of the following conditions hold:

- For all $j = 1, 2, \ldots, n$, $\lambda_j \in \mathbb{R}$;
- the trace of A is zero;
- the real parts of all eigenvalues are nonpositive;
- the real parts of all eigenvalues are nonnnegative.

The question remained opened at the end of the workshop in the case of real parts of mixed signs but the participants in this work group felt optimistic that a complete solution is within reach.

The same group also worked on finding the least positive t such that

$$\sigma = \{3 - t, 3 + t, -2, -2, -2, 0\}$$

is the spectrum of a 6×6 symmetric nonnegative matrix A. In the work of Laffey and Smigoc, it was found that $t = 1/3$ results into a realizable spectrum and it was conjectured that this is the minimum positive value for t. At the workshop, the problem was formulated as an optimization problem and it was confirmed numerically that this is indeed the case. Research continues for analogous cases with multiple 0's appearing in the spectrum.

Group 2

A group was formed mostly interested in eigenvalue problems arising in max algebra. One particular challenge reported is the following. One can define the characteristic polynomial of a matrix A under max algebra rules, using a max analogue of the permanent, as follows:

$$\mathcal{X}_A(\lambda) \;=\; \mathrm{per}\,(A \oplus \lambda \otimes I).$$

It was explained that $\mathcal{X}_A$ can be written as a standard polynomial in max algebra and factored (in almost linear time) into linear factors:

$$\prod_i^{\otimes} \;(\lambda \oplus a_i).$$

Such a polynomial is known to have only one root, namely $a_n = \lambda(A)$ (the max cycle mean). This sole eigenvalue plays the role of the Perron root. As it was put by Peter Butkovic, "what on earth are the other a_i's?", which are referred to as *corners*.

Group 3

This group considered a question motivated by a paper by Charles A. Micchelli and R. A. Willoughby: On functions which preserve the class of Stieltjes matrices. Linear Algebra Appl., 23:141-156, 1979. Let $A = sI - B$, where B is a symmetric matrix having the Perron-Frobenius property. Such a matrix can be viewed as a generalized form of an M-matrix and the question is what functions preserve this class of matrices. The group was able to show that absolutely monotonic functions have this preservation property. Another related question being considered regards products of factors $B - \mu_i I$, where μ_i's are eigenvalues of B, and whether or when such products also have the Perron-Frobenius property.

Group 4

A group of participants focused on qualitative questions regarding eventually nonnegative matrices. In particular, a characterization was pursued of sign patterns that allow eventual nonnegativity or eventual positivity. This group reported significant progress by presenting several sufficient or necessary conditions. Also some key patterns the allow or not allow eventual positivity were identified, resulting in several conjectures. Not surprisingly, the pattern of the positive entries plays a central role. The participants identified several other related questions, for example, what is the maximum number of negative entries that such a sign pattern may have, as well as questions about super-patterns and sub-patterns.

Group 5

Another group of participants worked on a perturbation problem proposed by Uri Rothblum. The main thrust was to consider a reducible nonnegative matrix A and its perturbed eigen-equation

$$(A + \epsilon E)v(\epsilon) = \rho(\epsilon)v(\epsilon),$$

where E is nonnegative, $\rho(\cdot)$ represents the spectral radius and $v(\cdot)$ a corresponding eigenvector. It is well-known that these perturbed quantities admit fractional power series expansions as follows:

$$\rho(\epsilon) = \sum_{i=0}^{\infty} \epsilon^{i/p}\, \rho_i \quad \text{and} \quad v(\epsilon) = \sum_{i=0}^{\infty} \epsilon^{i/p}\, v_i$$

for some ρ_i and v_i and some positive integer p. The problem posed is to determine the relationship between thee spectral properties of $A + \epsilon E$ and A. For example, what can one say about p based on (the eigenstructure) of A and conversely. The group reported that if A and E have some additional properties, then p is indeed the index of the spectral radius of A. They are also investigating the structure of the vectors v_i in the eigenvector expansion of $A + \epsilon E$. They expect that an intimate relationship exists between the vectors in a preferred basis for the generalized Perron eigenspace of A and the first p vectors of the Perron eigenvector of the power series expansion of $A + \epsilon E$.

Presentations

Danny Hershkowitz presented a review of results on the generalized eigenspace corresponding to the spectral radius of a nonnegative matrix, as well on how these results extend to general matrices. Tom Laffey surveyed the various inverse eigenvalue problems for nonnegative matrices, taking time to state their current status and questions crucial in taking the next step toward a solution. Peter Butkovic introduced the audience to max algebra and problems that parallel classical Perron-Frobenius theory. Stephan Gaubert delivered a general survey on non-linear Perron-Frobenius theory in finite dimensions, with some emphasis on game applications and techniques, and giving the tropical/maxplus perspective. Daniel Szyld's talk on eventually nonnegative matrices surveyed recent efforts in understanding their properties and applications. Hans Schneider compared the spectral theories of reducible nonnegative matrices in classical and max linear algebra. Pauline van den Driessche talked about the role of nonnegative and M-matrices in epidemiology. Several other participants also took the floor briefly for impromptu presentations during group discussions.

A Celebration

On Friday, December 6, we had the opportunity to surprise Tom Laffey with cake and wishes for a happy birthday!

Special Volume of ELA

As it was announced during the workshop, there will be a special volume of the Electronic Journal of Linear Algebra (ELA) on the occasion of the workshop.

Papers in nonnegative matrix theory inspired by the themes of the workshop are invited. These include, but are not limited to, spectral properties of nonnegative matrices and operators, inverse eigenvalue problems for nonnegative matrices, properties and patterns of eventually nonnegative matrices, cone and exponential nonnegativity, as well as matrices in max algebra.

Publication of papers will be immediate, following review and acceptance according to the standard policies of ELA. Papers should be prepared according to the journal guidelines found in

`http://www.math.technion.ac.il/iic/ela/`

and can be submitted to any one of the special editors:

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