Dichotomy Amenable/Nonamenable in Combinatorial Group Theory

organized by
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Workshop Summary

The notion of amenability was introduced by von Neumann in 1929 in his study of Banach-Tarski paradox [vN]. A topological group G is called amenable if there exists an invariant mean on the space of left uniformly continuous bounded functions on G. The notion of amenability has proved to be very rich and gradually spread over many domains in mathematics: harmonic analysis, representation theory, probability theory, group actions, geometry of metric spaces, ergodic theory, semi-simple Lie groups, etc. Amenability is studied for groups, semigroups, algebras, graphs, metric spaces, operator algebras, group actions, grupoids, foliations. The main goal of the workshop was to gain better understanding of the meaning of being amenable or non-amenable for a discrete, finitely generated group, through exploration of interesting and unexpected connections between group theory, dynamical systems, functional analysis, geometry, ergodic theory and probability observed in recent years by mathematicians working in different areas of mathematics. To this end we have invited to the workshop a host of experts with different background and different primary mathematical interests, but whose common point for this workshop was that amenability of discrete groups plays an important role in their research. It was planned that the participants of the workshop would concentrate their efforts on making progress towards four important open problems/conjectures:

(1) Unitarizability conjecture of Dixmier
(2) Benjamini-Schramm unicity of percolation conjecture
(3) Amenability of self-similar groups
(4) Amenability of Golod-Shafarevich groups

An important development occured several months prior to the workshop, namely, Laurent Bartholdi announced a proof of non-amenability of all Golod-Shafarevich groups [B2]. The positive result would have had interesting implications in representation theory and low-dimensional topology [Z]. The negative result is also very important: it gives a proof to a conjecture of Vershik from 70-ies, which states that the group is amenable if and only if the growth of ranks of quotients in its lower central series is subexponential. Bartholdi also proved that a group is amenable if and only if the group algebra is amenable [B1] (partial results in this direction were earlier obtained by Elek and by Gromov). The workshop featured an invited lecture by Bartholdi which was later followed by a discussion and by two working sessions. One session was held by four participants who went into the details of the proof of the main technical result of Barhtoldi's paper: adaptation to the setting of algebras of a tiling technique developed by B. Weiss for groups. The second session concentrated on possible use of non-amenability of Golod-Shafarevich groups to constructing a counter-example to Dixmier's unitarizability problem.

The first day of the workshop was devoted to the unitarizability problem of Dixmier. The greatest expert on this subject is Gilles Pisier [P], and the workshop started with his talk, in which he gave a very clear and "hands-on" introduction to the problem, explained what he (and other experts) have been able to achieve so far, and what major difficulties remain. After having tried to prove for many years that a group is amenable if and only if each of its uniformly bounded unitary representations is unitarizable, he now believes that a counterexample should exist, in the class of non-amenable groups without free subgroup. However he doesn't formulate this as a conjecture. In fact it is not known whether there exists a non-unitarizable group without free subgroups (such a group would be necessarily a counterexample to the von Neumann problem). It was proposed during Pisier's talk to consider one of the many known counterexamples to von Neumann's problem, say, the free Burnside groups and more general lacunary hyperbolic groups constructed in a recent paper by Olshanskii-Osin-Sapir [OOS]. Osin's first day talk gave an overview of these constructions. Work in this direction is still being carried on.

Pisier's talk was followed by a discussion session during which an idea was raised (it has been formulated earlier in the ICM 2006 talk by N. Monod [M]) that one way to prove that every non-amenable group is non-unitarizable (i.e., to give a positive answer to Dixmier's problem) is via measurable equivalence relations and the notion of randomorphisms. In order to be able to explore this idea, several activities were organized: a working session during which T. Tsankov and I. Epstein, PhD students of A. Khekris in Caltech and of G. Hjorth in UCLA, explained the notion of a "random" subgroup and associated equivalence relation. In a joint effort the strategy of proof was then laid out properly and technical difficulties have been discussed. This strategy is based on an unpublished result by D. Gaboriau and R. Lyons on existence of a "random" free subgroup in any non-amenable group. So a special session was organized the next day during which Lyons explained their result. It appears that the proof makes use of the characterization of amenable groups in terms of percolation (weak version of Benjamini-Schramm conjecture) due to Pak and Smirnova-Nagnibeda [PSN].

As a parallel activity, we also had a talk/problem session held by Guoliang Yu, in which he outlined other important problems on the intersection of group theory and functional analysis, including the "weak" amenability called Properties (A) and (L).

The second day of the workshop was devoted to percolation. Russell Lyons gave the introductory talk, and here again, was able to explain some techniques which most often remain completely out of the scope of conference plenary talks. In particular he pointed out, quite surprisingly, that Kazhdan's property (T) can be effectively used in proving things about percolation [LS]. A working session was held in the afternoon around new results of I. Kozakova, a PhD student of M. Sapir from Vanderbilt, on branching techniques in percolation on Cayley graphs [K, S].

The general idea of usefulness of the theory of measurable equivalence relations to the study of amenability and more generally, to the study of groups, was one of the main ideas of the workshop. Beside the above-mentioned discussions on "random" subgroups and on percolation, it also played a crucial role in the talk of Miklos Abert who exhibited its important connections to geometric topology [AN], and was surveyed in detail in the last talk of the workshop, by Vadim Kaimanovich.

Characterisation in terms of percolation is only one of several probabilistic approaches to amenability. Several discussions of these, triggered by the presence of such renown probabilists as Persi Diaconis and Yuval Peres, took place in the last two days of the workshop.

Diaconis opened the joint problem session by telling the audience of the Hunt-Stein theorem on estimation of risks in statistical decision problems. Amenability of a group acting on the underlying probability space happens to play an important role in the problem, which apparently was something completely new to everyone in the audience. The problem session was further chaired by Volodymyr Nekrashevych and featured problems which ranged from embeddings of groups in Hilbert spaces to existence of finitely generated amenable simple groups to Poisson boundary of infinite torsion groups.

Many specific questions were raised concerning amenability of self-similar groups. Earlier Nekrashevych gave an introductory talk on this subject in which he explained his new important step toward proving amenability: the absence of free subgroups. His talk was followed by a productive working session. In particular the related (but different!) notion of scale-invariant group, introduced by Benjamini, was discussed. Nekrashevych was able to reinterpret and to complete Gabor Pete's construction, which shows that lamplighter groups are scale-invariant (the problem raised by Benjamini was whether scale-invariance is equivalent to polynomial growth, see [S]). One more working session was held the next day, discussing self-similar random walks – a special technique developed by Bartholdi and Virag [BV] and later generalized by them together with Nekrashevych and Kaimanovich, used to prove amenability of one class of self-similar groups.

The talk of Yuval Peres revealed completely new connections between random walks on amenable groups and embeddings of groups in Hilbert spaces. He spoke of the inequality between the Hilbert space compression and the speed of the random walk and formulated a conjecture that it is in fact an equality. If true, this would imply in particular, that the speed of the random walk is a group invariant (independent on the choice of the generating set), which is a longstanding open problem. It would also imply another open conjecture (widely believed to be true), that a random walk on a Cayley graph cannot escape slower than in $\mathbb{Z}$. Balint Virag talked of an idea of A. Erschler about a possible way to prove this latter conjecture for finitely presented groups, by constructing a harmonic embedding in a Hilbert space. This became a subject of an active working session which resulted in a scheme of a proof.

According to our own impressions and to the feed-back we got from the workshop participants, the meeting was interesting and productive and has certainly contributed to the better understanding of the whole host of problems around amenability of discrete groups. The exchange of ideas was very intense during the workshop, and what was particularly precious, participants were able to discuss not only results, but also techniques leading to them, so one can reasonably expect some progress towards the open problems at the core of the workshop to happen as a result of the joint efforts of conference participants. Several working groups continue discussions after the end of a workshop. As far as we know, the discussions are related to the unitarizability, randomorphisms, Vershik's problem, speed of random walks on groups, percolation. One of the important results of the workshop is establishment of strong collaboration ties between people from different areas of mathematics: group theory, harmonic analysis, probability theory.

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