Stability Criteria for Multi-Dimensional Waves and Patterns

organized by
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Workshop Summary

1 Introduction

The meeting on Stability Criteria for Multi-Dimensional Waves and Patterns took place June 16-20, 2005, at the American Institute of Mathematics in Palo Alto, CA. The workshop was organized by Christopher K.R.T. Jones (University of North Carolina), Yuri Latushkin (University of Missouri), Robert Pego (Carnegie-Mellon University/University of Maryland), Bjorn Sandstede (Ohio State University/University of Surrey), and Arnd Scheel (University of Minnesota). The meeting focused on the study of stability properties of nonlinear waves, which are particular solutions to nonlinear partial differential equations. Dynamical systems theory provides powerful methods for the study of essentially one-dimensional waves, exploiting their description as heteroclinic and homoclinic orbits to a traveling-wave equation. For instance, the Evans function, a Wronskian type determinant which traces eigenvalues of the linearization at a nonlinear wave, has proven to be a fundamental tool for the stability analysis of waves in one space dimension, both from the theoretical and the numerical viewpoints. The Evans function factors in natural ways into sub-determinants when multiple spatial scales are involved, and winding number computations afford a numerically robust location of eigenvalues. However, the construction of the Evans function, as a determinant, is inherently finite-dimensional and an extension to higher space-dimensions would require an infinite-dimensional setting.

The workshop brought together researchers with backgrounds in functional analysis, dynamical systems, and numerical analysis, working on applications that cover, among others, pattern formation in biology, optical fibers, and water waves. With the expertise in Fredholm determinants, large-scale eigenvalue solvers, and ill-posed spatial dynamics, the goal of the workshop was to lay out a strategy for the analysis of the stability properties of genuinely higher-dimensional structures.

2 Overview of Meeting

The group consisted of 31 participants (4 graduate students, 3 postdocs, and 24 faculty) with backgrounds in applied PDEs and pattern formation, dynamical systems, viscous conservation laws, harmonic analysis, water waves, inverse scattering theory and numerical linear algebra.

The workshop began with three minicourses that were designed to outline the state of the art in multi-dimensional stability problems, thereby focusing the discussion on the issues central to the theme of the workshop. The entire group also discussed possible topics for breakout sessions that were then scheduled for Tuesday and Wednesday. The breakout sessions were stimulating and appeared to work very well. On Thursday, additional discussion sessions were organized in plenum: these were preceded by introductory presentations to set the scene and to set up topics for the following discussion. At this stage, various smaller groups also began to meet separately to discuss some of the issues that had arisen earlier in more depth.

The goal on Friday, the last day of the meeting, was to collect open problems, which were felt to be central to taking the field forward, from the participants and to organize them by formulating overarching themes. The format was that each participant had five minutes to present their problem, plus additional time for discussion: To facilitate discussion, we asked each contributor to put their problem on one of two flip-charts which were deemed doable and undoable problems! The necessity to explain their choice often provoked further constructive discussion.

3 Overarching Themes

3.1 Evans Function in Multidimensions

A. Four important points about the Evans function:

(i) Easier to compute if there is a singular perturbation structure that can be exploited

(ii) Parity calculation: relate structure of the wave to calculation of parity

(iii) Eigenvalues produced out of the essential spectrum

(iv) Relation to resolvent estimates and poles

B. Strategies to constructing the Evans function in multiD:

(i) Work with structures that are multidimensional but retain some 1D structure that can be exploited in the calculations.

(ii) Another completely different strategy is to take a functional analytic approach for the definition of the Evans function, namely by the use of Fredholm determinants which are easily defined in multiD. One might be able to use the ideas of Latushkin and write the determinant as the identity plus something compact (a sandwiched resolvent?)

(iii) Sweeping of multidimensions by domain dilations. An example of this is the recent work of Deng and Jones.

C. Issues to be resolved for multiD Evans function:

(i) What relation of geometric information can be encoded in the Evans function

(ii) Do we really need the Evans function to be analytic?

(iii) Transient dynamics and time dependent solutions described as hidden objects

D. Categorizing the plethora of problems needing an multiD Evans function

(i) Origin of the original problem: Another strategy to defining the Evans function is to categorize the problems to which the Evans function arises (ie viscous conservation laws, dispersive equations etc). It is likely that different tools will be helpful in each context.

(ii) compact versus noncompact regions

(iii) Finite Galerkin modes vs not

3.2 Numerical Computation of the Evans Function

A. Four important points to consider:

(i) Use of exterior powers vs. individual vectors,

(ii) Nyquist diagrams,

(iii) Continuation methods,

(iv) Low rank approximations to far field conditions.

4 Conclusions and Impact

A great deal of information was shared. Many participants mentioned that they learned a lot from the workshop, indeed a surprising amount given the unconventional format. What emerged over the course of the week was, on the one hand, a comprehensive retrospective on the remarkable progress achieved in the subject in the last few years, and, on the other hand, a systematic assessment of future prospects. In particular, the factors that have contributed to success in treating one-dimensional problems with Evans functions were delineated and analyzed with a view toward what is needed for multi-dimensional problems. Participants also reviewed and discussed the latest results related to extending the use of exponential dichotomies to multi-dimensions and connecting Evans functions to tools such as modified Fredholm determinants. Numerous problems for future work were suggested, classified, and focused upon, covering a great variety of nonlinear wave models, including reaction-diffusion systems, nonlinear Schrdinger equations, plasma dynamics, conservation laws, combustion, water waves, and more.

New collaborations were started, and new ideas were generated for ways to carry out stability analyses, and especially underlying computations and numerical analysis. Cross fertilization between analysis and computation worked in both directions. Some particular ideas that arose involve the use of bordered matrices for eigenvalue analysis, and the idea that Rouchs theorems from analytic operator theory may be brought to bear on problems of numerical analysis.

5 Future Directions

The participants of the conference identified the following directions for future research and some particularly important problems:

Study of the following multidimensional formations in nonlinear waves: (a) spirals; (b) radially symmetric patterns; (c) defects; (d) hot spots;

Study of applications of Fredholm determinants in multidimensional problems and other connections to functional analytic techniques including the Gohberg-Sigal-Rouche

Theorem and connections of the Krein signature to the Evans function

Further study of edge bifurcations and absolute spectra for one dimensional traveling waves including dynamical interpretation of roots of the Evans functions imbeded in the absolute spectrum;

Further study of the linearized and nonlinear stability/instability for travelling waves in Kadomtsev-Petriashvili and Benjamin-Ono equations;

Study of stability of strong shocks in the multidimensional setting and multidimensional questions arising in fluid dynamics and MHD equations;

Study of bifurcations and dynamics of localized stripes in reaction-diffusion systems;

Extensions of the Evans functions to co-dimension 1 fronts and moving fronts which change profile when they evolve;

Scattering dynamics for dissipative particles;

Further development of the blow-up techniques for generalized KdV;

Study of the best numerical integration schemes and other practical computational methods for constructing the Evans function;

Study of the Evans function in 2- or 3-dimensional models such as Gross-Pitaevsky equation with parabolic potentials;

Stability and instability for non-planar waves in dispersive and dissipative systems, and systems of conservation laws; further applications of the Evans function in proving stability results;

Applications to biological problems;

Synthesis of numerical and functional-analytic methods in stability for travelling waves;

Study of plane waves, shock waves, and stability in kinetic and Boltzmann equations, and stability of travelling waves in the full water problem;

Study of the spatial stability for Toda lattices and other discrete models;

Study of the limiting cases in the generalized KdV equations when there is a lost of exponential dichotomy;

Detailed study of the front dynamics, in particular, for returned fronts;

Further development of topological invariants, such as the Maslov Index, in higher dimensions.

Study of combined structures in viscous conservation laws.