Motivic Donaldson-Thomas theory and singularity theory

organized by
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Workshop Summary

The focus of this workshop was interactions between Donaldson-Thomas theory and singularity theory. In recent years, it has become increasingly clear that there are interesting relations between these active but quite-different subjects. A great deal of recent progress in Donaldson-Thomas theory - an enumerative theory of sheaves on threefolds - has come from its interpretation (due to Behrend) in terms of constructions in singularity theory. This insight has allowed the application of a new toolkit of techniques and also suggests the possibility of enhanced invariants (motivic, categorical, etc.). In parallel, singularity theorists have been studying related problems about characteristic classes associated to singular hypersurfaces. Our goal was to bring together researchers from different perspectives, which so far have remained largely disjoint, and to promote cross-pollination of ideas, questions, and techniques. We feel that the workshop was a success at fostering these interactions.

The workshop was organized according to the standard AIM format, as follows. Each morning there were two lectures and in the afternoon, we collected problem suggestions and split into small groups to work on them. The first afternoon was devoted to an "ask-the-expert" question session, where we split participants into two groups (for each subject) moderated by an expert from the other side. On the last afternoon, we collected a list of open problems in the subject to focus subsequent activity in the subject.

The morning lectures were designed to survey major topics in the field and, as the week progressed, to discuss recent progress. On the first day, we had a talk by Balázs Szendröi surveying the themes of the conference, and a talk by Kai Behrend introducing his work relating virtual fundamental classes symmetric obstruction theories to Chern-Macpherson characteristic classes. On the second day, the theme of the lectures was to discuss motivic refinements of Donaldson-Thomas invariants. The key idea here is the motivic Milnor fiber of an algebraic function, a lift of the original topological notion. We started with a talk by François Loeser, explaining this construction. Maxim Kontsevich gave the second talk surveying his groundbreaking paper with Yan Soibelman in which they introduce a program to define motivic Donaldson-Thomas invariants for any three-dimensional Calabi-Yau category satisfying certain assumptions. One key point in this program is the existence of a coherent choice of orientation assigned to such a category, and this was discussed throughout the week during afternoon sessions. On Wednesday, the focus moved to cohomological refinements of the Milnor number - either via the sheaf of vanishing cycles or via a twisted version of the de Rham complex. Claude Sabbah gave an introductory talk on twisted de Rham complex and its relation to vanishing cycles. Yan Soibelman explained work with Kontsevich explaining how in certain cases one can use these cohomological refinements to define an algebraic structure (the cohomological Hall algebra) which again encodes the usual

DT invariants. One problem that arises here of interest to many participants is under what conditions these sheaf-theoretic constructions can be glued together globally.

On Thursday, we had talks by Laurentiu Maxim surveying ideas related to characteristic classes of singular hypersurfaces. We also had a talk by Vivek Shende explaining singularities of plane curves to DT-type invariants - while this topic is not directly related to previous ones, it is another overlap between enumerative geometry and singularity theory. On Friday, Dominic Joyce gave a talk on shifted symplectic structures with potential applications to categorifying DT invariants. He surveyed the recent notion of Pantev-Toen-Vaquie-Vezzosi of a shifted symplectic form on a derived scheme; this is a more complicated but better-behaved notion than the symmetric obstruction theories defined by Behrend. The hope is that this structure (with orientation data) will enable us to glue together the local constructions discussed Wednesday (vanishing cycles, etc.). Finally, by request, we concluded the workshop with a talk by Alex Dimca who explained some Hodge-theoretic aspects of the twisted de Rham complex.

There were a wide variety of problems discussed during the afternoon sessions. The questions suggested during moderated problem sessions were in directions that could be profitably studied by both DT and singularity-theorists; the emphasis varied from concrete goals to more open-ended speculative questions. On the concrete side, topics included investigating possible counterexamples to the existence of orientation data in geometric setting and calculating the value of the Behrend function for the Hilbert scheme of points on affine three-space. One promising avenue studied during the week was studying DT invariants for the Calabi-Yau category of local systems on a real orientable three-manifold. Another direction involved certain basic questions involving non-isolated critical loci, e.g. the possibility of a localization formalism for studying vanishing cohomology, or to what extent a function can be recovered from its critical locus. Other topics discussed included possible applications to geometric engineering, the relevance of higher Chern classes, and wall-crossing formalism for enumerative invariants of quiver Grassmannians. While none of these questions were definitively answered, many of the groups had partial progress to report after only a few days, and we expect that the work started here will continute fruitfully in the months to come. Finally, on Friday, we had a session to list what we viewed as interesting or outstanding problems for upcoming progress in these subjects.

Overall, we felt that the workshop was a good first step towards many of our goals. In addition to making some progress on short-term questions, there was a lot of interaction between the two communities in terms of educating each side about the research interests of the other. We expect that this should yield research dividends in the immediate future in terms of increased collaboration. We are very grateful to AIM, NSF, and the Renyi Institute for the support and the hospitality.